STABLE HOMOLOGY OF AUTOMORPHISM GROUPS OF FREE GROUPS

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ABSTRACT. Homology of the group $\mathrm{Aut}(F_n)$ of automorphisms of a free group on n generators is known to be independent of n in a certain stable range. Using tools from homotopy theory, we prove that in this range it agrees with homology of symmetric groups. In particular we confirm the conjecture ([HV98b]) that stable rational homology of $\mathrm{Aut}(F_n)$ vanishes.

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1. Introduction

1.1. **Results.** Let $F_n = \langle x_1, \dots, x_n\rangle$ be the free group on n generators and let $\mathrm{Aut}(F_n)$ be its automorphism group. Let Σ_n be the symmetric group and let $\varphi_n : \Sigma_n \to \mathrm{Aut}(F_n)$ be the homomorphism that to a permutation σ associates the automorphism $\varphi(\sigma) : x_i \mapsto x_{\sigma(i)}$. The main result of the paper is the following theorem.

Theorem 1.1. *φ_n induces an isomorphism*

$$(\varphi_n)_*\colon H_k(\Sigma_n) \to H_k(\mathrm{Aut}(F_n))$$

for $n > 2k+1$.

The homology groups in the theorem are independent of n in the sense that increasing n induces isomorphisms $H_k(\Sigma_n) \cong H_k(\Sigma_{n+1})$ and $H_k(\mathrm{Aut}(F_n)) \cong H_k(\mathrm{Aut}(F_{n+1}))$ when $n > 2k+1$. For the symmetric group this was proved by Nakaoka ([Nak60]) and for $\mathrm{Aut}(F_n)$ by Hatcher and Vogtmann ([HV98a],[HV04]). The homology groups $H_k(\Sigma_n)$ are completely known. With finite coefficients the calculation was done by Nakaoka and can be found in [Nak61]. We will not quote the result here. With rational coefficients the homology groups vanish because Σ_n is a finite group, so theorem 1.1 has the following corollary.

Corollary 1.2. *The groups*

$$H_k(\mathrm{Aut}(F_n); \mathbb{Q})$$

vanish for $n > 2k+1$.

The groups $\mathrm{Aut}(F_n)$ are special cases of a more general series of groups A_n^s, studied in [HV04] and [HVW06]. We recall the definition. For a finite graph G without vertices of valence 0 and 2, let ∂G denote the set of vertices of valence 1. Let $h\mathrm{Aut}(G)$ denote the topological monoid of homotopy equivalences $G \to G$ that restrict to the identity map on ∂G. Let $\mathrm{Aut}(G) = \pi_0 h\mathrm{Aut}(G)$.

Definition 1.3. Let G_n^s be a connected graph with s leaves and first Betti number $b_1(G_n^s) = n$. For $s + n \geq 2$ let

$$A_n^s = \mathrm{Aut}(G_n^s).$$

In particular $A_n^0 = \mathrm{Out}(F_n)$ and $A_n^1 = \mathrm{Aut}(F_n)$. A_0^s is the trivial group for all s.

There are natural group maps for $n \geq 0$, $s \geq 1$

$$A_n^{s-1} \xleftarrow{\alpha_n^s} A_n^s \xrightarrow{\beta_n^s} A_n^{s+1} \xrightarrow{\gamma_n^s} A_{n+1}^s. \tag{1.1}$$

β_n^s and γ_n^s are induced by gluing a Y-shaped graph to G_n^s along part of ∂G_n^s. α_n^s is induced by collapsing a leaf. We quote the following theorem.

Theorem 1.4 ([HV04],[HVW06]). *$(\beta_n^s)_*$ and $(\gamma_n^s)_*$ are isomorphisms for $n > 2k+1$. $(\alpha_n^s)_*$ is an isomorphism for $n > 2k+1$ for $s > 1$ and $(\alpha_n^1)_*$ is an isomorphism for $n > 2k+3$.*

The main theorem 1.1 calculates the homology of these groups in the range in which it is independent of n and s. In other words, we calculate the homology of the group

$$\mathrm{Aut}_\infty = \operatorname*{colim}_{n\to\infty} \mathrm{Aut}(F_n).$$

An equivalent formulation of the main theorem is that the map of classifying spaces $B\Sigma_\infty \to B\mathrm{Aut}_\infty$ is a *homology equivalence*, i.e. that the induced map in integral homology is an isomorphism. The Barratt-Priddy-Quillen theorem ([BP72]) gives a homology equivalence $\mathbb{Z} \times B\Sigma_\infty \to QS^0$, where QS^0 is the infinite loop space

$$QS^0 = \operatorname*{colim}_{n\to\infty} \Omega^n S^n.$$

The main theorem 1.1 now takes the following equivalent form.

Theorem 1.5. *There is a homology equivalence*

$$\mathbb{Z} \times B\mathrm{Aut}_\infty \to QS^0.$$

Alternatively the result can be phrased as a homotopy equivalence $\mathbb{Z} \times B\mathrm{Aut}_\infty^+ \simeq QS^0$, where $B\mathrm{Aut}_\infty^+$ denotes Quillen's plus-construction applied to $B\mathrm{Aut}_\infty$. Quillen's plus-construction converts homology equivalences to weak homotopy equivalences, cf. e.g. [Ber82].

Most of the theorems stated or quoted above for $\mathrm{Aut}(F_n)$ have analogues for mapping class groups. Theorem 1.4 above is the analogue of the homological stability theorems of Harer and Ivanov for the mapping class group ([Har85], [Iva89]). Corollary 1.2 above is the analogue of "Mumford's conjecture", and the homotopy theoretic strengthening in theorem 1.5 (which is equivalent to the statement in theorem 1.1) is the analogue of Madsen-Weiss' generalized Mumford conjecture ([MW02], see also [GMTW06]).

Some conjectures and partial results in this direction have been known. Hatcher ([Hat95]) noticed that there is a homotopy equivalence $\mathbb{Z} \times B\mathrm{Aut}_\infty^+ \simeq QS^0 \times W$ for some space W. Hatcher and Vogtmann [HV98b] calculated $H_k(\mathrm{Aut}(F_n);\mathbb{Q})$ for small k. They proved that $H_4(\mathrm{Aut}(F_4);\mathbb{Q}) = \mathbb{Q}$ and that $H_k(\mathrm{Aut}(F_n);\mathbb{Q}) = 0$ for all other (k,n) with $0 < k \leq 6$. It follows that the stable rational homology vanishes

in degrees ≤ 6, and they conjectured that stable rational cohomology vanishes in all degrees. Corollary 1.2 verifies Hatcher-Vogtmann's conjecture.

1.2. **Outline of proof.** Culler-Vogtmann's *Outer Space* plays the role for $\mathrm{Out}(F_n)$ that Teichmüller space plays for mapping class groups. Since its introduction in [CV86], it has been of central importance in the field, and firmly connects $\mathrm{Out}(F_n)$ to the study of *graphs*. A point in outer space X_n is given by a triple (G, g, h) where G is a connected finite graph, g is a metric on G, i.e. a function from the set of edges to $[0, \infty)$ satisfying that the sum of lengths of edges in any cycle of G is positive, and h is a marking, i.e. a conjugacy class of an isomorphism $\pi_1(G) \to F_n$. Two triples (G, g, h) and (G', g', h') define the same point in X_n if there is an isometry $\varphi : G \to G'$ compatible with h and h'. The isometry is allowed to collapse edges in G of length 0 to vertices in G'. If G has N edges, the space of metrics on G is an open subset $M(G) \subseteq [0, \infty)^N$. Equip $M(G)$ with the subspace topology and X_n with the quotient topology from $\amalg M(G)$, the disjoint union over all marked graphs (G, h). This defines a topology on X_n and Culler-Vogtmann proves that it is contractible.

Outer space is built using compact connected graphs G with fixed first Betti number $b_1(G) = n$. The main new tool in this paper is the definition of a space $\Phi(\mathbb{R}^N)$ of *non-compact* graphs $G \subseteq \mathbb{R}^N$. Inside the space $\Phi(\mathbb{R}^N)$ is a space B_N of embedded compact graphs. We will prove that a connected component of B_∞ is weakly equivalent to $B\mathrm{Out}(F_n)$. Considering also non-compact graphs allows us to define a map

$$B_N \xrightarrow{\tau_N} \Omega^N \Phi(\mathbb{R}^N). \tag{1.2}$$

In the analogy to mapping class groups, τ_N replaces the Pontryagin-Thom collapse map of [MW02] and [GMTW06] and as N varies, the spaces $\Phi(\mathbb{R}^N)$ form a spectrum $\mathbf{\Phi}$ which replaces the Thom spectrum $MTO(d)$ of [GMTW06] and $\mathbb{C}P^\infty_{-1}$ of [MW02]. We take the direct limit of (1.2) as $N \to \infty$ and get a map

$$B_\infty \xrightarrow{\tau_\infty} \Omega^\infty \mathbf{\Phi}$$

or, by restriction to a connected component,

$$B\mathrm{Out}(F_n) \to \Omega^\infty \mathbf{\Phi}.$$

Composing with the map induced by the group maps $\alpha^1_{n+1} \circ (\gamma^1_n \circ \beta^1_n) : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1}) \to \mathrm{Out}(F_{n+1})$ from (1.1) we get a map

$$\coprod_{n\geq 0} B\mathrm{Aut}(F_n) \to \Omega^\infty \mathbf{\Phi}, \tag{1.3}$$

and the proof of theorem 1.5 is concluded in the following steps.

(i) The map (1.3) induces a well defined $\tau : \mathbb{Z} \times B\mathrm{Aut}_\infty \to \Omega^\infty \mathbf{\Phi}$,
(ii) τ is a homology equivalence,
(iii) $\Omega^\infty \mathbf{\Phi} \simeq QS^0$.

The paper is organized as follows. In chapter 2 we define and study the space $\Phi(\mathbb{R}^N)$. In chapter 3 we define a subspace $B_N \subseteq \Phi(\mathbb{R}^N)$ consisting of compact graphs and explain its relation to $B\mathrm{Out}(F_n)$. We also define the map (1.2). The proof that there is an induced map $\tau : \mathbb{Z} \times B\mathrm{Aut}_\infty \to \Omega^\infty \mathbf{\Phi}$ which is a homology equivalence is in chapter 4 and is in two steps. First, in section 4.1 we define a topological category $\mathcal{C}$, whose objects are finite sets and whose morphisms are certain *graph cobordisms*. We prove the equivalence $\Omega B\mathcal{C} \simeq \Omega^\infty \mathbf{\Phi}$. Secondly, in section 4.2, we prove that there is a homology equivalence $\mathbb{Z} \times B\mathrm{Aut}_\infty \to \Omega B\mathcal{C}$. This is very similar to, and inspired by, the corresponding statements for mapping class groups in [GMTW06]. Finally, chapter 5 is devoted to proving that $\Omega^\infty \mathbf{\Phi} \simeq QS^0$. This completes the proof of theorem 1.5.

In the supplementary chapter 6 we compare with the work in [GMTW06]. Our proof of theorem 1.5 works with minor modifications if the space $\Phi(\mathbb{R}^N)$ is replaced throughout by a space $\Psi_d(\mathbb{R}^N)$ of smooth d-manifolds $M \subseteq \mathbb{R}^N$ which are closed subsets. In that case we prove an unstable version of the main result of [GMTW06]. To explain it, let $\mathrm{Gr}_d(\mathbb{R}^N)$ be the Grassmannian of d-planes in $\mathbb{R}^N$, and $U^\perp_{d,N}$ the canonical $(N-d)$ dimensional vector bundle over it. Let $\mathrm{Th}(U^\perp_{d,N})$ be its Thom space. Then we prove the weak equivalence

$$B\mathcal{C}^N_d \simeq \Omega^{N-1}\mathrm{Th}(U^\perp_{d,N}), \tag{1.4}$$

where $\mathcal{C}^N_d$ is now the cobordism category whose objects are closed $(d-1)$-manifolds $M \subseteq \{a\} \times \mathbb{R}^{N-1}$ and whose morphisms are compact d-manifolds $W \subseteq [a_0, a_1] \times \mathbb{R}^{N-1}$, cf. [GMTW06, section 2]. In the limit $N \to \infty$ we recover the main theorem of [GMTW06], but (1.4) holds also for finite N.

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2. The sheaf of graphs

This chapter defines and studies a certain sheaf Φ on $\mathbb{R}^N$. Roughly speaking, $\Phi(U)$ will be the set of all graphs $G \subseteq U$. We allow non-compact, and possibly infinite, graphs. The precise definition is given in section 2.1 below, where we also define a topology on $\Phi(U)$, making Φ a sheaf of topological spaces.

2.1. Definitions.

Recall that a continuous map $f : X \to Y$ is a *topological embedding* if $X \to f(X)$ is a homeomorphism, when $f(X) \subseteq Y$ has the subspace topology. If X an Y are smooth manifolds, then f is a C^1 embedding if f is C^1, if $Df(x) : T_xX \to T_{f(x)}Y$ is injective for all $x \in X$, and if f is a topological embedding.

Definition 2.1. Let $U \subseteq \mathbb{R}^N$ be open. Let $\Phi(U)$ be the set of pairs (G, l), where $G \subseteq U$ is a subset and $l : G \to [0, 1]$ is a continuous function. (G, l) is required to satisfy the following properties.

(i) If $p \in U - G$, there is a neighborhood $U_p \subseteq U$ of p with $G \cap U_p = \emptyset$.
(ii) If $p \in l^{-1}([0, 1))$, there is a neighborhood $U_p \subseteq U$ of p, an open set $V \subseteq (-1, 1)$, and an C^1 embedding $\gamma : V \to U_p$ such that $G \cap U_p = \gamma(V)$ and $l \circ \gamma(t) = t^2$.
(iii) If $p \in l^{-1}(1)$, there is a neighborhood $U_p \subseteq U$ of p, a natural number $n \geq 3$, an open set $V \subseteq \vee^n[-1, 1)$, and a topological embedding $\gamma : V \to U_p$ such that $G \cap U_p = \gamma(V)$ and $l \circ \gamma(t) = t^2$. We require γ to be C^1 in the following sense. Let $j_k : [-1, 1) \to \vee^n[-1, 1)$ be the inclusion of the kth wedge summand. Then $\gamma_k = \gamma \circ j_k : (j_k)^{-1}(V_p) \to U_p$ is C^1, and the vectors $\gamma_k'(0)/|\gamma_k'(0)|$ are pairwise different, $k = 1, \dots, n$.

A *graph in U* is an element $G \in \Phi(U)$.

(i) is equivalent to $G \subseteq U$ being a closed subset. G need not be compact, and non-compact $G \in \Phi(U)$ may have infinitely many edges and vertices. G also need not be connected.

Let us say that a parametrizations $\gamma : V \to G$ as in (ii), with $V \subseteq (-1, 1)$ and satisfying $l \circ \gamma(t) = t^2$, is an *admissible parametrization of G at p*. These are almost unique: If $\bar\gamma$ is another admissible parametrization at p, then either $\gamma(t) = \bar\gamma(t)$ or $\gamma(t) = \bar\gamma(-t)$ for t near $\gamma^{-1}(p)$. So specifying the function $l : G \to [0, 1]$ is equivalent to specifying an equivalence class $\{[\gamma], [\bar\gamma]\}$ of germs of parametrizations around each point. Often we will omit l from the notation and write e.g. $G \in \Phi(U)$.

An inclusion $U \subseteq U'$ induces a restriction map $\Phi(U') \to \Phi(U)$ given by

$$G \mapsto G \cap U.$$

This makes Φ a sheaf on $\mathbb{R}^N$. More generally, if $j : U \to U'$ is an embedding (not necessarily an inclusion) of open subsets of $\mathbb{R}^N$, define $j^* : \Phi(U') \to \Phi(U)$ by

$$j^*(G) = j^{-1}(G)$$

and $j^*(l) = l \circ j : j^*(G) \to [0, 1]$.

We have the following standard terminology.

Definition 2.2. Let $G \in \Phi(U)$.

(i) Let $\mathscr{V}(G) = l^{-1}(1)$. This is the set of *vertices* of G.
(ii) An *edge point* is a point in the 1-manifold $\mathscr{E}(G) = G - \mathscr{V}(G)$.
(iii) An *oriented edge* is a continuous map $\gamma : [-1, 1] \to G$ such that $l \circ \gamma(t) = t^2$ and such that $\gamma|(-1, 1)$ is an embedding.
(iv) A *closed edge* of G is a subset $I \subseteq G$ which is the image of some oriented edge. Each edge is the image of precisely two oriented edges. If I is the image of γ, it is also the image of the oriented edge given by $\bar{\gamma}(t) = \gamma(-t)$.
(v) A subset $T \subseteq G$ is a *tree* if it is the union of finitely many vertices and closed edges of G and if T is contractible.

The following notion of maps between elements of $\Phi(U)$ is important for defining the topology on $\Phi(U)$. We remark that it does not make $\Phi(U)$ into a category (because composition is only partially defined).

Definition 2.3. Let $G', G \in \Phi(U)$. A morphism $\varphi : G' \dashrightarrow G$ is a triple (V', V, φ), where $V' \subseteq G'$ and $V \subseteq G$ are open subsets and $\varphi : V' \to V$ is a continuous surjection satisfying the following two conditions.

(i) For each $v \in \mathscr{V}(G) \cap V$, $\varphi^{-1}(v) \subseteq G'$ is a tree. Let $\mathscr{V}(V) = \mathscr{V}(G) \cap V$, $\mathscr{E}(V) = V - \mathscr{V}(V)$,
$$\mathscr{V}(\varphi) = \bigcup_{v \in \mathscr{V}(V)} \varphi^{-1}(v)$$
and $\mathscr{E}(\varphi) = V' - \mathscr{V}(\varphi)$.
(ii) φ restricts to a C^1 diffeomorphism of manifolds over $[0, 1)$
$$\mathscr{E}(\varphi) \to \mathscr{E}(V). \tag{2.1}$$

Throughout the paper we will use dashed arrows for partially defined maps. Thus the notation $f : X \dashrightarrow Y$ means that f is a function $f : U \to X$ for some subset $U \subseteq X$.

It can be seen in the following way that for any morphism (V', V, φ) as above, the underlying map of spaces $\varphi : V' \to V$ is proper. Let $K \subseteq V$ be compact, and let $x_n \in \varphi^{-1}(K)$ be a sequence of points. After passing to a subsequence we can assume that the sequence $y_n = \varphi(x_n)$ converges to a point $y \in K$. If $y \in \mathscr{E}(V)$ we must have $x_n \to x$ with $x = \varphi^{-1}(y) \in \varphi^{-1}(K)$ because (2.1) is a diffeomorphism. If $y \in \mathscr{V}(V)$ then $T = \varphi^{-1}(y) \subseteq G$ is a tree and it is immediate from the definitions that a tree has a compact neighborhood $C \subseteq V'$ with $C - T \subseteq \mathscr{E}(\varphi)$. It follows that $C = \varphi^{-1}(\varphi(C))$ and that $\varphi(C) \subseteq V$ is a neighborhood of V. Therefore $x_n \in C$ eventually, and hence x_n has a convergent subsequence.

Definition 2.4. Let $\varepsilon > 0$. Let $K \subseteq U$ be compact and write K^ε for the set of $k \in K$ with $\operatorname{dist}(k, U - K) \geq \varepsilon$.

(i) $\varphi = (V', V, \varphi)$ is (ε, K)-small if $K \cap G \subseteq V$, $K^\varepsilon \cap G' \subseteq \varphi^{-1}(K)$, and if

$$|k - \varphi(k)| < \varepsilon \quad \text{for all } k \in \varphi^{-1}(K).$$

We point out that $\varphi^{-1}(K) \subseteq V'$ is a compact set containing $K^\varepsilon \cap G'$.

(ii) If $Q \subseteq K - \mathscr{V}(G)$ is compact, then φ is (ε, K, Q)-small if it is (ε, K)-small and if for all $q \in Q \cap G$ there is an admissible parametrization γ with $q = \gamma(t)$ and

$$|(\varphi^{-1} \circ \gamma)'(t) - \gamma'(t)| < \varepsilon.$$

(iii) For ε, K, Q as above, let $\mathscr{U}_{\varepsilon,K,Q}(G)$ be the set

$$\{G' \in \Phi(U) \mid \text{there exists an } (\varepsilon, K, Q)\text{-small } \varphi : G' \dashrightarrow G\}.$$

For the case $Q = \emptyset$ we write $\mathscr{U}_{\varepsilon,K}(G) = \mathscr{U}_{\varepsilon,K,\emptyset}(G)$.

(iv) The C^0-topology on $\Phi(U)$ is the topology generated by the set

$$\{\mathscr{U}_{\varepsilon,K}(G) \mid G \in \Phi(U), \varepsilon > 0,\ K \subseteq U \text{ compact}\}. \tag{2.2}$$

(v) The C^1-topology on $\Phi(U)$ is the topology generated by the set

$$\{\mathscr{U}_{\varepsilon,K,Q}(G) \mid G \in \Phi(U), \varepsilon > 0,\ K \subseteq U \text{ and } Q \subseteq K - \mathscr{V}(G) \text{ compact}\}. \tag{2.3}$$

Unless explicitly stated otherwise, we topologize $\Phi(U)$ using the C^1-topology. In lemma 2.8 below we prove that the sets (2.2) and (2.3) form bases for the topologies they generate, and that the sets $\mathscr{U}_{\varepsilon,K}(G)$ form a neighborhood basis at G in the C^0-topology for fixed G and varying $\varepsilon > 0$, $K \subseteq U$ and similarly $\mathscr{U}_{\varepsilon,K,Q}(G)$ in the C^1 topology.

Example 2.5. We discuss the important example $G = \emptyset \in \Phi(U)$, which will illustrate the role of the compact set K. Any morphism $(V', V, \varphi) : G' \dashrightarrow \emptyset$ must have $V' = V = \emptyset$, because $V \subseteq G = \emptyset$ and $\varphi : V' \to V$. Thus φ is (ε, K)-small if and only if $K^\varepsilon \cap G' = \emptyset$. In particular

- If X is a topological space then a map $f : X \to \Phi(U)$ is continuous at a point $x \in X$ with $f(x) = \emptyset$ if and only if for all compact subsets $K \subseteq U$ there exists a neighborhood $V \subseteq U$ of x such that $f(y) \cap K = \emptyset$ for all $y \in V$.
- If $(G_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\Phi(U)$, then $G_n \to \emptyset$ if and only if for all compact subsets $K \subseteq U$ there exists $N \in \mathbb{N}$ such that $G_n \cap K = \emptyset$ for $n > N$.

Lemma 2.6. *The space $\Phi(\mathbb{R}^N)$ is path connected.*

Proof. We construct an explicit path from a given $G \in \Phi(\mathbb{R}^N)$ to the basepoint $\emptyset \in \Phi(\mathbb{R}^N)$. Choose a point $p \in \mathbb{R}^N - G$ and let $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$, $t \in [0, 1]$ be the map given by

$$\varphi_t(x) = (1 - t)x + tp.$$

Then φ_t is a diffeomorphism for $t < 1$ and $\varphi_1(x) = p$ for all x. Let $G_t = (\varphi_t)^{-1}(G)$. This defines a map $t \mapsto G_t \in \Phi(\mathbb{R}^N)$. We will see later (lemma 2.11) that it is continuous on $[0, 1)$. Continuity at 1 can be seen as follows. For a given compact $K \subseteq \mathbb{R}^N$, choose $\delta > 0$ such that $K \subseteq B(p, \delta^{-1})$. Then $G_t \cap K = \emptyset$ for $t > 1 - \delta$. □

2.2. **Point-set topological properties.** In this section we prove various results about $\Phi(U)$ of a point-set topological nature. The verifications are elementary, but somewhat tedious, and their proofs could perhaps be skipped at a first reading.

Let us first point out that V in definition 2.4 can always be made smaller: If $(V', V, \varphi) : G' \dashrightarrow G$ is (ε, K, Q)-small, then (W', W, ψ) is (ε, K, Q)-small if $K \cap G \subseteq W \subseteq V$ with $W \subseteq V$ open, $W' = \varphi^{-1}(W)$, and $\psi = \varphi|W'$.

Lemma 2.7. *If $(V', V, \varphi) : G' \dashrightarrow G$ is (ε, K, Q)-small and $(V'', W', \psi) : G'' \to G'$ is (δ, K', Q')-small with $K' \supseteq \varphi^{-1}(K) \cup K^\varepsilon$ and $Q' \supseteq \varphi^{-1}(Q)$, then $(V'', V, \varphi \circ \psi)$ is an $(\varepsilon + \delta, K)$-small morphism after possibly shrinking V and W'.*

Proof. It suffices to consider the case $K' = \varphi^{-1}K \cup K^\varepsilon$. We have $\varphi^{-1}(K) \subseteq K' \cap G' \subseteq W'$ by assumption on K' and by (δ, K')-smallness of (V'', W', ψ). Therefore the subset $\varphi(V' - W') \subseteq V$ is disjoint from K, and properness of $\varphi : V' \to V$ implies that $\varphi(V' - W') \subseteq V$ is

closed. After replacing V by $V - \varphi(V' - W')$ we can assume that $V' = \varphi^{-1}(V) \subseteq W'$.

We have $K^\varepsilon \cap G' \subseteq \varphi^{-1}(K) \subseteq V'$ because (V', V, φ) is (ε, K)-small, so $K' \cap G' \subseteq (K^\varepsilon \cup \varphi^{-1}(K)) \cap G' \subseteq V'$. Hence after shrinking W' we can assume that $W' = V'$. Then $(V'', V, \varphi \circ \psi)$ is a morphism. It is $(\varepsilon + \delta, K, Q)$-small because $K^{\varepsilon+\delta} \cap G'' \subseteq (\varphi \circ \psi)^{-1}(K)$ and for $k \in (\varphi \circ \psi)^{-1}(K) \subseteq \psi^{-1}(K')$ we have

$$|k - \varphi \circ \psi(k)| \leq |k - \psi(k)| + |\psi(k) - \varphi(\psi(k))| < \delta + \varepsilon.$$

A similar condition on first derivatives holds on $(\varphi \circ \psi)^{-1}(Q)$. $\square$

Lemma 2.8. *Let $G \in \Phi(U)$, $\varepsilon > 0$ and $K \subseteq U$ and $Q \subseteq K - \mathscr{V}(G)$ compact. Let $G' \in \mathscr{U}_{\varepsilon,K,Q}(G)$. Then there exists $\delta > 0$ and compact $K' \subseteq U$, $Q' \subseteq K' - \mathscr{V}(G')$ such that*

$$\mathscr{U}_{\delta,K',Q'}(G') \subseteq \mathscr{U}_{\varepsilon,K,Q}(G). \tag{2.4}$$

We can take $Q' = \emptyset$ if $Q = \emptyset$.

Proof. Let $(V', V, \varphi) : G' \dashrightarrow G$ be (ε, K, Q)-small. By compactness of $\varphi^{-1}(K)$, we can choose $\delta > 0$ with $|\varphi(k) - k| < \varepsilon - \delta$ for all $k \in \varphi^{-1}(K)$. By compactness of Q we can assume that

$$|(\varphi^{-1} \circ \gamma)'(t) - \gamma'(t)| < \varepsilon - \delta.$$

for all admissible parametrizations γ of G with $\gamma(t) = q \in Q$. We can also assume that δ satisfies $\delta < \mathrm{dist}(K^\varepsilon, \varphi^{-1}(G - \mathrm{int}(K))$. Then (V', V, φ) is actually $(\varepsilon - \delta, K, Q)$-small, and the claim follows from lemma 2.7 if we set

$$K' = \varphi^{-1}(K) \cup K^{(\varepsilon-\delta)}, \quad Q' = \varphi^{-1}(Q). \qquad \square$$

Lemma 2.8 implies that the set (2.3) is a basis for the topology it generates, and that the collection of $\mathscr{U}_{\varepsilon,K,Q}(G)$ forms a neighborhood basis at G, for fixed G and varying ε, K, Q. Similarly for the C^0-topology.

The next lemma is the main rationale for including the map $l : G \to [0,1]$ into the data of an element of $\Phi(U)$. It gives a partial uniqueness result for the (ε, K)-small maps $(V', V, \varphi) : G' \dashrightarrow G$, whose existence is assumed when $G' \in \mathscr{U}_{\varepsilon,K}(G)$.

Lemma 2.9. *For each $G \in \Phi(U)$ and each compact $C \subseteq U$, there exists an $\varepsilon > 0$ and a compact $K \subseteq U$ with $C \subseteq \mathrm{int}(K^\varepsilon)$ such that for $G' \in \mathscr{U}_{\varepsilon,K}(G)$, any two (ε, K)-small maps*

$$\varphi, \psi : G' \dashrightarrow G$$

must have $\varphi = \psi$ near C.

Notice that both ψ and φ are defined near $C\cap G'$ when $C \subseteq \operatorname{int}(K^\varepsilon)$.

Proof. Let $W \subseteq U$ be an open set with compact closure $\overline{W} \subseteq U$ and $C \subseteq W$. We will prove that ε and K can be chosen so that $\varphi^{-1}(w) = \psi^{-1}(w)$ for all $w \in W$ when both maps are (ε, K)-small. If we also arrange $\varepsilon < \operatorname{dist}(C, U - W)$, that will prove the statement in the lemma.

First take $\varepsilon > 0$ and $K \subseteq U$ such that $W \subseteq K^{2\varepsilon}$. We can assume that the distance between any two elements of $K \cap \mathscr{V}(G)$ is greater than 2ε. Then the triangle inequality implies that $\varphi^{-1}(v) = \psi^{-1}(v)$ for all $v \in K^{2\varepsilon} \cap \mathscr{V}(G)$. It remains to treat edge points.

Let $M \subseteq G \cap \operatorname{int}(K)$ be the smallest open and closed subset containing $G \cap W$. We claim that $\varphi^{-1}(v) = \psi^{-1}(v)$ for v in $M - \mathscr{V}(G)$. It suffices to consider $v \in M \cap l^{-1}((0,1))$ since that set is dense in $M - \mathscr{V}(G)$ (we omit only "midpoints" of edges). Compactness of $\overline{W}$ implies that $\pi_0 M$ is finite (connected components of $G \cap \operatorname{int}(K)$ are open in $G \cap \operatorname{int}(K)$, so the compact subset $\overline{W} \cap G$ can be non-disjoint from only finitely many). $l^{-1}(\{0,1\}) \cap K$ is a finite set of points, so $\pi_0(M \cap l^{-1}((0,1)))$ is also finite. Choose a $\tau > 0$ such that the inclusion

$$W \cap l^{-1}([\tau, 1-\tau]) \to M \cap l^{-1}((0,1)) \tag{2.5}$$

is a π_0-surjection (i.e. the induced map on π_0 is surjective).

The function $l' : G' \to [0,1]$ restricts to a local diffeomorphism $(l')^{-1}((0,1)) \to (0,1)$. It follows that the diagonal embedding

$$(l')^{-1}((0,1)) \xrightarrow{\text{diag}} \{(k,m) \in G' \times G' \mid l(k) = l(m) \in (0,1)\} \tag{2.6}$$

has open image. Therefore (by continuity of $\psi^{-1}, \varphi^{-1} : l^{-1}((0,1)) \to G'$) the set

$$\{k \in l^{-1}((0,1)) \cap M \mid \psi^{-1}(k) = \varphi^{-1}(k)\}$$

is open and closed in $l^{-1}((0,1)) \cap M$, so it suffices to prove that it contains a point in each path component of $l^{-1}((0,1)) \cap M$. We prove that $\psi^{-1} = \varphi^{-1}$ when composed with the π_0-surjection (2.5).

The set of pairs (k,m) with $k, m \in K \cap l^{-1}([\tau, 1-\tau])$, and $l(k) = l(m)$ and $k \neq m$ is a compact subset of $K \times K$, so we can assume that $|k - m| > 2\varepsilon$ for such (k,m). Now let $k \in W \cap l^{-1}([\tau, 1-\tau]) \in K^{2\varepsilon} \cap G'$, assume $\psi^{-1}(k) \neq \varphi^{-1}(k)$, and set $x = \psi^{-1}(k)$. ψ is (ε, K)-small, so $|x - k| = |x - \psi(x)| < \varepsilon$. Hence $x \in K^\varepsilon \cap G'$, so $\varphi(x)$ is defined and $\varphi(x) \in G \cap G$. Injectivity of φ (on non-collapsed edges) implies that $\varphi(x) = \varphi(\psi^{-1}(k)) \neq k = \psi(x)$. Set $m = \varphi(x)$. Since $l(k) = l'(x) = l(m) \in [\tau, 1-\tau]$, we have

$$2\varepsilon < |k - m| = |\psi(x) - \varphi(x)| \leq |\psi(x) - x| + |x - \varphi(x)|$$

which contradicts φ and ψ being (ε, K)-small. $\square$

Proposition 2.10. *Φ is a sheaf of topological spaces on $\mathbb{R}^N$, i.e. the following diagram is an equalizer diagram of topological spaces for each covering of U by open sets U_j, $j \in J$*

$$\Phi(U) \to \prod_{j\in J} \Phi(U_j) \rightrightarrows \prod_{(i,l)\in J\times J} \Phi(U_i \cap U_l).$$

Proof. Let $V \subseteq U$ be open and let $r : \Phi(U) \to \Phi(V)$ denote the restriction map. If $G \in \Phi(V)$ and $G' \in r^{-1}\mathscr{U}_{\varepsilon,K,Q}(G)$, lemma 2.8 provides $\delta > 0$ and $K', Q' \subseteq V$ such that $\mathscr{U}_{\delta,K',Q'}(rG') \subseteq \mathscr{U}_{\varepsilon,K,Q}(G)$. If $\operatorname{dist}(K, \mathbb{R}^N - V) > \delta$ we have

$$\mathscr{U}_{\delta,K',Q'}(G') = r^{-1}\mathscr{U}_{\delta,K',Q'}(rG') \subseteq r^{-1}\mathscr{U}_{\varepsilon,K,Q}(G)$$

which proves that $r^{-1}\mathscr{U}_{\varepsilon,K,Q}(G)$ is open and hence that r is continuous. Therefore the maps in the diagram are all continuous. The proposition holds for both the C^0 and the C^1 topology. We treat the C^0 case first.

Let $\tilde{\Phi}(U)$ denote the image of $\Phi(U) \to \prod \Phi(U_i)$, topologized as a subspace of the product. Then $\Phi(U) \to \tilde{\Phi}(U)$ is a continuous bijection. Take $G \in \Phi(U)$ and $\varepsilon > 0$ and let $K \subseteq U$ be compact. We will prove that the image of $\mathscr{U}_{\varepsilon,K}(G) \subseteq \Phi(U)$ in $\tilde{\Phi}(U)$ is a neighborhood of the image $\tilde{G} \in \tilde{\Phi}(U)$ of $G \in \Phi(U)$.

Choose a finite subset $\{j_1, \dots, j_n\} \subseteq J$ and compact $C_i \subseteq U_{j_i}$ such that $K \subseteq \cup_{i=1}^n C_i^\delta$ for some $\delta \in (0, \varepsilon)$. Let $K_i \subseteq U_{j_i}$ be compact subsets with $C_i \subseteq \operatorname{int}(K_i)$. Let $K_{il} = K_i \cap K_l$. By lemma 2.9 we can assume, after possibly shrinking δ and enlarging the K_i, that (δ, K_{il})-small morphisms $\varphi_{il} : G_{il} \dashrightarrow (G|U_{j_i j_l})$ with $G_{il} \in \Phi(U_{il})$ have unique restriction to a neighborhood of $G \cap C_{il}$. Thus, if $G' \in \Phi(U)$ has a (δ, K_i)-small $\varphi_i : (G'|U_{j_i}) \dashrightarrow (G|U_{j_i})$ for all $i = 1, \dots, n$, then φ_i and φ_l agrees near $G \cap C_{il}$. Therefore they glue to a morphism $\varphi : G' \dashrightarrow G$ which is defined near $L = \cup_i C_i$ and agrees with φ_i near C_i. Since φ_i is (δ, K_i)-small we will have $\varphi_i(C_i) \supseteq C_i^\delta \cap G$ and hence $\varphi(L) \supseteq K \cap G$ so the image of φ contains $K \cap G$. The domain contains $L \cap G' = \cup_{i=1}^n (C_i \cap G')$ which contains $G \cap G'$ and hence $K^\delta \cap G'$. Finally, let $k \in G$ have $\varphi(k) \in K$ and hence $\varphi(k) C_i^\delta \subseteq K_i$ for some i. Then $\varphi(k) = \varphi_i(k)$ and

$$|\varphi(k) - k| = |\varphi_i(k) - k| < \delta$$

because φ_i is (δ, K_i)-small. We get that φ is (δ, K)-small.

We have proved that $G' \in \mathscr{U}_{\delta,K}(G) \subseteq \mathscr{U}_{\varepsilon,K}(G)$ whenever $(G'|U_{j_i}) \in \mathscr{U}_{\delta,K_i}(G|U_{j_i})$ for each $i = 1, \dots, n$. Therefore the image of $\mathscr{U}_{\varepsilon,K}(G) \subseteq$

$\Phi(U)$ in $\tilde{\Phi}(U)$ contains $p^{-1}(\mathscr{U})$, where

$$\mathscr{U} = \prod_{i=1}^{n} \mathscr{U}_{\delta,K_i}(G|U_{j_i}) \subseteq \prod_{i=1}^{n} \Phi(U_{j_i}), \tag{2.7}$$

and p is the projection $p : \tilde{\Phi}(U) \to \prod_{i=1}^{n} \Phi(U_i)$. $p^{-1}(\mathscr{U})$ is the required neighborhood of $\tilde{G}$.

To prove the C^1 case, suppose $Q \subseteq K - \mathscr{V}(G)$, repeat the proof of the C^0 case, and set $Q_i = K_i \cap Q$. Then replace $\mathscr{U}_{\delta,K_i}$ by $\mathscr{U}_{\delta,K_i,Q_i}$ in (2.7). ☐

The sheaf property implies that continuity of a map $f : X \to \Phi(U)$ can be checked locally in $X \times U$. In other words, f is continuous if for each $x \in X$ and $u \in U$ there is a neighborhood $V_x \times W_u \subseteq X \times U$ such that the composition

$$V_x \to X \xrightarrow{f} \Phi(U) \xrightarrow{\text{restr.}} \Phi(W_u)$$

is continuous. In particular, $U \mapsto \operatorname{Map}(X, \Phi(U))$ is a sheaf for every space X.

Proposition 2.11. *If $V \subseteq U$ are open subsets of $\mathbb{R}^N$, then the restriction map $\Phi(U) \to \Phi(V)$ is continuous. More generally, the action map*

$$\operatorname{Emb}(V, U) \times \Phi(U) \to \Phi(V)$$

$(j, G) \mapsto j^*(G)$ is continuous, where $\operatorname{Emb}(V, U)$ is given the C^1 topology.

Proof sketch. Let $j \in \operatorname{Emb}(V, U)$, $G \in \Phi(U)$ and let $\varepsilon > 0$, and let $K \subseteq V$ be compact. Choose $\delta > 0$ and compact subsets $C \subseteq V$ and $L \subseteq jV$ such that $K \subseteq C^\delta$ and $jK \subseteq L^\delta$. Choose a number M such that

$$|j^{-1}(l) - j^{-1}(l')| \leq M|l - l'| \quad \text{and} \quad |D_l j^{-1}(v)| \leq M|v|$$

for all $l, l' \in L$ and $v \in T_l(U)$. We can assume $2M\delta \leq \varepsilon$. Then $j^*\varphi : j^*G' \to j^*G$ is $(\varepsilon/2, K)$-small if $\varphi : G' \to G$ is (δ, K)-small.

Let $j' \in \operatorname{Emb}(V, U)$ be another embedding such that $j^{-1} \circ j' : V \dashrightarrow V$ is $(\varepsilon/2, K')$-small in the sense that the domain contains $(K')^{\varepsilon/2}$, the image contains K', and that $|f(k) - k| < \varepsilon/2$ for $f(k) \in K'$. Then the composition

$$(j')^*G' \xrightarrow{j^{-1} \circ j'} j^*(G') \xrightarrow{j^*\varphi} j^*G$$

is (ε, K)-small by lemma 2.7, provided $K' \supseteq (j^*\varphi)^{-1}(K) \cup K^\varepsilon$, which is satisfied if $(K')^\varepsilon \supseteq K$. This proves continuity when $\Phi(U)$ and $\Phi(V)$ are given the C^0 topology. The C^1 topology is similar. □

3. Homotopy types of graph spaces

Lemma 2.6 shows that the full space $\Phi(\mathbb{R}^N)$ is path connected. A similar argument shows that $\Phi(\mathbb{R}^N)$ is in fact $(N-3)$-connected. In this section we study the homotopy types of certain subspaces of $\Phi(\mathbb{R}^N)$.

3.1. Graphs in compact sets.

Definition 3.1. (i) For a closed subset $A \subseteq U$, let $\Phi(A)$ be the set of germs around A, i.e. the colimit of $\Phi(V)$ over open sets with $A \subseteq V \subseteq U$. We remark that the colimit topology is often not well behaved (for example if A is a point then the one-point subset $\{[\emptyset]\} \subseteq \Phi(A)$ is dense), and we consider $\Phi(A)$ as a set only.

(ii) Let $U \subseteq \mathbb{R}^N$ be open and $M \subseteq U$ compact. For a germ $S \in \Phi(U - \mathrm{int} M)$, let $\Phi^S(M)$ be the inverse image of S under the restriction $\Phi(U) \to \Phi(U - \mathrm{int} M)$. Topologize $\Phi^S(M)$ as a subspace of $\Phi(U)$.

(iii) Let $G', G \in \Phi^S(M)$. A *graph epimorphism* $G' \to G$ is a morphism (V', V, φ) in the sense of definition 2.3 which is surjective and everywhere defined (i.e. $V' = G'$ and $V = G$). Furthermore φ is required to restrict to the identity map $S \to S$.

(iv) Let $\mathcal{G}_S$ be the category with objects $\Phi^S(M)$ and graph epimorphisms as morphisms. We consider $\mathrm{ob}(\mathcal{G}_S)$ and $\mathrm{mor}(\mathcal{G}_S)$ discrete sets.

The main result in this section is the following theorem.

Theorem 3.2. *Let $U \subseteq \mathbb{R}^N$ be open and $M \subseteq U$ compact. Let $S \in \Phi(U - \mathrm{int} M)$. Assume $\mathrm{int} M$ is $(N-3)$-connected. Then there is an $(N-3)$-connected map*

$$\Phi^S(M) \to B\mathcal{G}_S.$$

In the next section we prove that the classifying space $B\mathcal{G}_S$ is homotopy equivalent to a space built out the spaces BA_n^s, where A_n^s are the groups from theorem 1.4. Combined with theorem 3.2 above this leads to theorem 3.19, which summarizes the results of sections 3.1, 3.2, and 3.3. We need more definitions for the proof.

Definition 3.3. (i) Let $M \subseteq U$ be compact and $R \in \Phi(U)$. Let $S = [R] \in \Phi(U - \mathrm{int} M)$ be the germ of R. Let $\Phi(M; R)$ be the set of pairs (G, f), where $G \in \Phi^S(M)$ and $f : G \to R$ is a graph epimorphism.

(ii) Let $(G, f) \in \Phi(M; R)$. For (ε, K, Q) as in definition 2.4, let $\mathscr{U}_{\varepsilon,K,Q}(G, f) \subseteq \Phi(M; R)$ be the set of (G', f') which admits an (ε, K, Q)-small $\varphi : G' \dashrightarrow G$ with $f' = \varphi \circ f$. Topologize $\Phi(M; R)$ by declaring that the $\mathscr{U}_{\varepsilon,K,Q}(G, f)$ form a basis.

(iii) Let $\mathrm{Emb}_R(M) \subseteq \Phi(M; R)$ be the subspace in which the morphism $f : G \to R$ has an inverse morphism $f^{-1} : R \to G$.

The space $\mathrm{Emb}_R(M)$ can be thought of as a space of certain embeddings $R \to U$. Namely $(G, f) \in \mathrm{Emb}_R(M)$ can be identified with the map $f^{-1} : R \to G \subseteq U$.

Throughout the paper we will make extensive use of simplicial spaces. Recall that a simplicial space $X_\bullet$ has a geometric realization $\|X_\bullet\|$ and that a simplicial map $f_\bullet : X_\bullet \to Y_\bullet$ induces $\|f_\bullet\| : \|X_\bullet\| \to \|Y_\bullet\|$. If for each k the map $f_k : X_k \to Y_k$ is $(n-k)$-connected, the geometric realization $\|f_\bullet\|$ is n-connected. In particular $\|f_\bullet\|$ is a weak equivalence if each f_k is a weak equivalence. Recall also that to each category C is an associated classifying space BC, defined as the geometric realization of the nerve $N_\bullet C$.

If C is a category (usually not topologized) and $F : C \to \text{Spaces}$ is a functor, then the *homotopy colimit* of F is defined as

$$\operatorname*{hocolim}_{C} F = B(C \wr F)$$

where $(C \wr F)$ is the category whose objects are pairs (c, x) with $c \in \mathrm{ob}(C)$ and $x \in F(c)$, and whose morphisms $(c, x) \to (c', x')$ is the set of morphisms $f \in C(c, c')$ with $F(f)(x) = x'$. If $T : F \to G$ is a natural transformation such that $T(x) : F(x) \to G(x)$ is n-connected for each object x, the induced map $\operatorname{hocolim} F \to \operatorname{hocolim} G$ is also n-connected.

The proof of theorem 3.2 is broken down into the following assertions, whose proofs occupy the remainder of this section and the following.

- The forgetful map
$$\operatorname*{hocolim}_{R \in \mathcal{G}_S} \Phi(M; R) \to \Phi^S(M)$$
induced by the projection $(G, f) \mapsto G$ is a weak equivalence.
- The inclusion $\mathrm{Emb}_R(M) \to \Phi(M; R)$ is a weak equivalence.
- The space $\mathrm{Emb}_R(M)$ is $(N-4)$-connected if $\mathrm{int}(M)$ is $(N-3)$-connected.

The following lemma will be used again throughout the paper. Recall that a map is *etale* if it is a local homeomorphism and an open map.

Lemma 3.4. *Let C be a topological category and Y a space. Regard Y as a category with only identity morphisms, and let $f : C \to Y$ be a*

functor such that $N_0 f$ and $N_1 f$ are etale maps. Assume that $B(f^{-1}(y))$ is contractible for all $y \in Y$. Then $Bf : BC \to Y$ is a weak equivalence.

Proof sketch. The hypothesis implies that a neighborhood of the fiber

$$B(f^{-1}(y)) \subseteq BC$$

is homeomorphic, as a space over Y, to a neighborhood of

$$\{y\} \times B(f^{-1}(y)) \subseteq Y \times B(f^{-1}(y)).$$

Then the result follows from [Seg78, proposition (A.1)]. □

Lemma 3.5. *The forgetful map $p : \Phi(M; R) \to \Phi^S(M)$, $p(G, f) = G$, is etale.*

Proof. Let $(G, f) \in \Phi(M; R)$. An application of lemma 2.9 gives an $\varepsilon > 0$ and a compact $K \subseteq U$ such that any $G' \in \Phi^S(M) \cap \mathscr{U}_{\varepsilon,K}(G)$ will have a *unique* graph epimorphism $\varphi_{G'} : G' \to G$ which is (ε, K)-small. $\varphi_{G'}$ restricts to the identity outside M. This gives a map $G' \mapsto (G', f \circ \varphi_{G'})$ which is a local inverse to p. We have proved that p restricts to a homeomorphism

$$\mathscr{U}_{\varepsilon,K}(G, f) \to \mathscr{U}_{\varepsilon,K}(G).$$ □

Proposition 3.6. *The map*

$$\operatorname*{hocolim}_{R \in \mathcal{G}_S} \Phi(M; R) \to \Phi^S(M)$$

induced by the projection $(G, f) \mapsto G$ is a weak equivalence.

Proof. The maps from lemma 3.5 assemble to a map

$$\coprod_{R \in \mathcal{G}_S} \Phi(M; R) \xrightarrow{p} \Phi^S(M).$$

The domain of this map is the space of objects of the category $(\mathcal{G}_S \wr \Phi(M; -))$. Morphisms $(R', (G', f')) \to (R, (G, f))$ exist only if $G' = G$; then they are morphisms $\varphi : R' \to R$ in $\mathcal{G}_S$ with $\varphi \circ f' = f$. The classifying space of this category is the homotopy colimit in the proposition, and p induces a map

$$Bp : \operatorname*{hocolim}_{R \in \mathcal{G}_S} \Phi(M; R) \to \Phi^S(M).$$

Let $G \in \Phi^S(M)$. Then the subcategory $p^{-1}(G) \subseteq (\mathcal{G}_S \wr \Phi(M; -))$ has $(G, (G, \mathrm{id}))$ as initial object. Therefore $(Bp)^{-1}(G) = B(p^{-1}(G))$ is contractible, so p satisfies the hypotheses of lemma 3.4. □

3.2. **Spaces of graph embeddings.** Our next aim is to prove that the space $\Phi(M;R)$ is highly connected when $\mathrm{int}M \subseteq \mathbb{R}^N$ is highly connected and N is large. The main step is to prove that the inclusion $\mathrm{Emb}_R(M) \to \Phi(M;R)$ is a weak equivalence. Although it is slightly lengthy to give all details, the idea is easy to explain. Suppose $(G,f) \in \Phi(M;R)$, and we want to construct a path to an element in $\mathrm{Emb}_R(M)$. The map $f : G \to R$ specifies a finite set of trees $T_v = f^{-1}(v) \subseteq G$, $v \in \mathscr{V}(R) \cap \mathrm{int}M$, such that G becomes isomorphic to R when every $T_v \subseteq G$ is collapsed to a point. The point is that this contraction can be carried out inside M, by continuously shortening leaves of the tree T_v and "dragging along" edges incident to T_v (see the illustration in figure 2). The formal proof consists of making this construction precise and proving it can be done continuously. The construction is remotely similar to the Alexander trick.

We begin by constructing a prototype collapse. This is done in construction 3.8 below, illustrated in figure 2.

Definition 3.7. Let $(G,l) \subseteq \Phi(\mathbb{R}^N)$, and let $T \subseteq G$ be a tree. An *incident edge* to T is a map $\gamma : [0,\tau] \to G$ with $\tau < 2$ such that $l(\gamma(t)) = (t-1)^2$ and $\gamma^{-1}(T) = \{0\}$. We consider two incident edges equivalent if one is a restriction of the other. Say that (G,T) is in *collapsible position* if all $g \in G \cap B(0,3)$ are in either T or in the image of an incident edge, if $T \subseteq \mathrm{int}D^N$, and if there are representatives $\gamma_i : [0,\tau_i] \to G$ for all the incident edges satisfying

$$\begin{aligned} |\gamma_i(\tau_i)| &> 3, \\ \langle \gamma_i(t), \gamma_i'(t) \rangle &\geq 0, \quad \text{when } |\gamma_i(t)| \in [1,3]. \end{aligned}$$

These γ_i provide a "distance to T" function $d : G \cap B(0,3) \to [0,2)$ given by $d(x) = 0$ when $x \in T$ and $d(\gamma_i(t)) = t$.

We point out that if $G \in \Phi(\mathbb{R}^N)$ and if there exists a T with (G,T) in collapsible position, then T is unique (it must be the union of all closed edges of G contained in $\mathrm{int}B(0,1)$), and the function $d : G \cap B(0,3) \to [0,2)$ is independent of choice of representatives γ_i.

Construction 3.8. Let $\lambda_{\frac{1}{3}} : [0,\infty) \to [0,\infty)$ be a smooth function satisfying $\lambda_{\frac{1}{3}}(r) = 3r/2$ for $r \leq 1.3$, $\lambda_{\frac{1}{3}}(r) = 2$ for $1.4 \leq r \leq 1.9$, and $\lambda_{\frac{1}{3}}(r) = r$ for $r > 2.5$. We also assume $\lambda'_{\frac{1}{3}}(r) \geq 0$ and $\lambda'_{\frac{1}{3}}(r) > 0$ for $\lambda_{\frac{1}{3}}(r) \neq 2$ and $\lambda'_{\frac{1}{3}}(r) \leq r^{-1}\lambda_{\frac{1}{3}}(r)$. For $t \in [0, \frac{1}{3}]$, let

$$\lambda_t(r) = (1-3t)r + 3t\lambda_{\frac{1}{3}}(r)$$

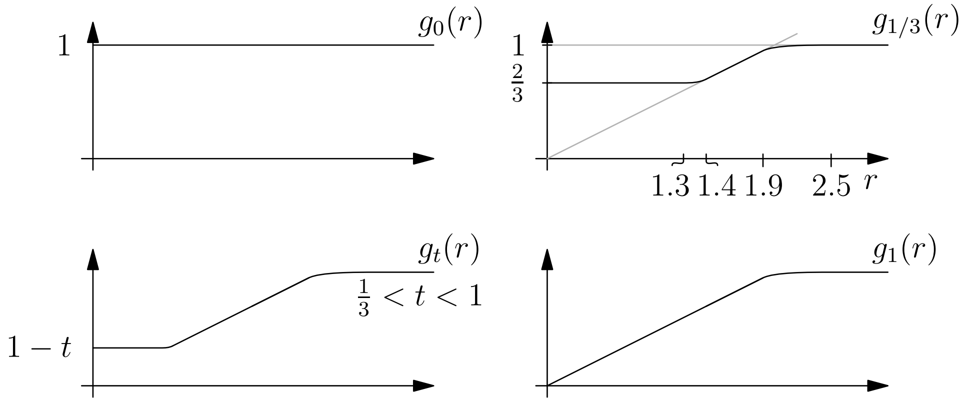

FIGURE 1. g_t for various $t \in [0,1]$.

and for $(t,r) \in [\frac{1}{3},1] \times [0,\infty) - \{(1,0)\}$ let

$$\lambda_t(r) = \begin{cases} \lambda_{\frac{1}{3}}(\frac{2r}{3(1-t)}) & r \leq 1.5(1-t) \\ 2 & 1.5(1-t) \leq r \leq 1.9 \\ \lambda_{\frac{1}{3}}(r) & r \geq 1.9. \end{cases}$$

Let $g_t(r) = (\lambda_t(r))^{-1}r$ and $g_t(0) = 1$ for $t \leq 0$ and $g_t(0) = (1-t)$ for $0 \leq t \leq 1$. The graph of g_t is shown in figure 1 for various values of $t \in [0,1]$. Define $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\varphi_t(x) = \frac{x}{g_t(|x|)}.$$

Thus φ_t multiplies by $(t-1)^{-1}$ near $\varphi_t^{-1}(B(0,1))$ and is the identity outside $\varphi_t^{-1}(B(0,2.5))$. φ_t preserves lines through the origin and $|\varphi_t(x)| = \lambda_t(|x|)$, so the critical values of φ_t when $t \geq 1/3$ are precisely the points in $2S^{N-1}$. We leave $\varphi_1(0)$ undefined.

(i) For $T \subseteq G \in \Phi(\mathbb{R}^N)$ in collapsible position, define a path of subsets $G_t \subseteq \mathbb{R}^N$ by

$$G_t = \begin{cases} \varphi_t^{-1}(G) & \text{for } t < 1, \\ \{0\} \cup \varphi_1^{-1}(G) & \text{for } t = 1. \end{cases}$$

(ii) For $x \in \varphi_t^{-1}(T)$ or $|\varphi_t(x)| \geq 3$, let $l_t(x) = l(\varphi_t(x))$.

(iii) If $x \in G_t$ has $\varphi_t(x) \in B(0,3) - T$, define $l_t(x) = (d_t(x)-1)^2$, where d_t is defined as

$$d_t(x) = g_t(|x|)d(\varphi_t(x)) \tag{3.1}$$

and $d : G \cap B(0,3) \to [0,2)$ is the function from definition 3.7.

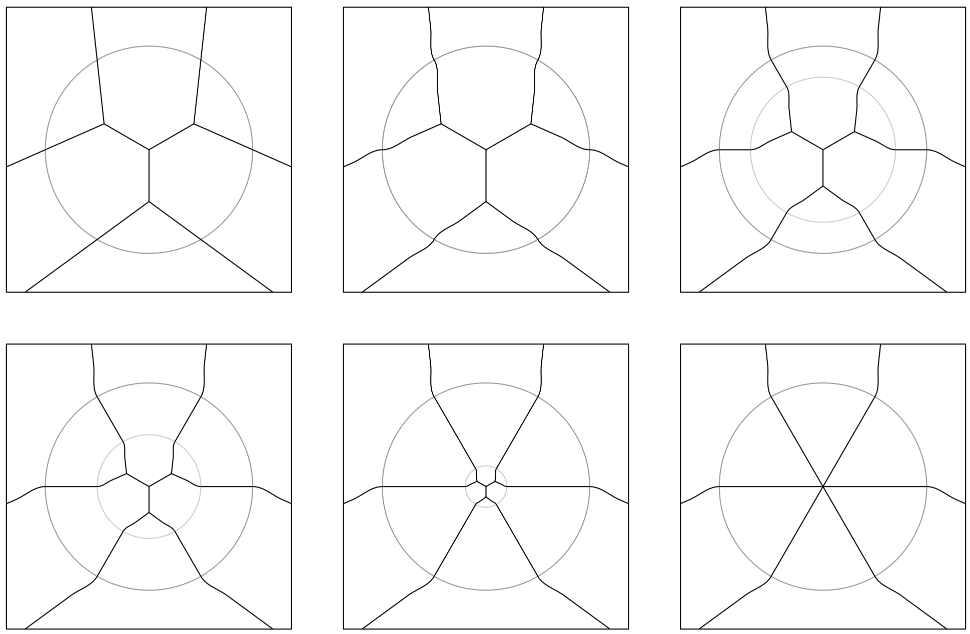

FIGURE 2. G_t for various $t \in [0,1]$, cf. construction 3.8.

The collapse of a tree $T \subseteq G$ in collapsible position in construction 3.8 above is illustrated in figure 2. The outer gray circle in each picture is $\partial B(0,2)$, and the region between the two gray circles is $\varphi_t^{-1}(\partial B(0,2))$.

Lemma 3.9. *For (G,T) in collapsible position, the above construction gives elements $\Upsilon_t(G,l) = (G_t, l_t) \in \Phi(\mathbb{R}^N)$ for all $t \in [0,1]$. Moreover, $(t,(G,l)) \mapsto (G_t, l_t)$ defines a continuous map $\Upsilon : [0,1] \times C \to \Phi(\mathbb{R}^N)$, where $C \subseteq \Phi(\mathbb{R}^N)$ is the open subspace consisting of graphs in collapsible position.*

Proof. The assumptions on G imply that the set $\{g \in G \mid 1 \leq |g| \leq 3\}$ contains no vertices of G and no critical points of the function $g \mapsto |g|$. Therefore $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$ is transverse to the edges of G and for each vertex v of g, φ_t is a diffeomorphism near $\varphi_t^{-1}(v)$. This implies that G_t satisfies the requirements of definition 2.1, except possibly that parametrizations γ satisfy $l \circ \gamma(t) = t^2$.

Let $\gamma_t : (a,b) \to G_t$ be a parametrization of an edge of G_t such that $\varphi_t \circ \gamma_t(s)$ maps to the image of an incident edge, and $|\gamma_t(s)|$ is an increasing function of s. Then a direct calculation shows that $d_t(\gamma_t(s))$ has strictly positive derivative with respect to s. Indeed, in formula (3.1), both factors $g_t(|\gamma_t(s)|)$ and $d_t(\varphi_t(x))$ have non-negative derivative, and at least one of them has strictly positive derivative. On the subset

$\varphi_t^{-1}(G \cap \partial B(0,2)) \subseteq G_t$ (the region between the gray circles in figure 2) the factor $d(\varphi_t(\gamma_t(s)))$ is constant and the factor $g_t(|\gamma_t(s)|)$ is $2|\gamma_t(s)|$. After reparametrizing γ_t we can assume $d_t(\gamma_t(s)) = s+1$ in which case γ_t is an admissible parametrization of G_t. This implies that $(G_t, l_t) \in \Phi(\mathbb{R}^N)$ for all $t \in [0,1]$.

Thus to each incident edge $\gamma : [0,\tau] \to G = G_0$ corresponds an admissible parametrization $\gamma_t : [0,\tau] \to \varphi_t^{-1}(\mathrm{Im}(\gamma))$ and $\gamma_{t'}$ and γ_t have images that are diffeomorphic via the map $x \mapsto \gamma_{t'} \circ d_t(x)$. These assemble to a map $G_t \to G_{t'}$ which is an isomorphism of graphs for $t, t' < 1$. For $t' = 1$ they assemble to a graph epimorphism $G_t \to G_1$ given by $x \mapsto \gamma_1 \circ d_t(x)$ outside $\varphi_t^{-1}(T)$ and collapsing $\varphi_t^{-1}(T) \subseteq G_t$ to $0 \in G_1$.

To prove continuity of $(t,(G,l)) \mapsto (G_t, l_t)$, let $t \in [0,1]$, $G \in C$, $u \in \mathbb{R}^N$. We prove continuity at each point $(t,(G,l),u) \in [0,1] \times C \times \mathbb{R}^N$ (cf. proposition 2.10 and the remark following its proof). For $(t,u) \neq (1,0)$, continuity follows from the implicit function theorem, and it remains to prove continuity at $(1,G) \in [0,1] \times C$ at $0 \in R^N$. This follows from the above mentioned graph epimorphism $G_t \to G_1$, because $\varphi_t^{-1}(T) \subseteq B(0,1-t)$. □

Notice also that admissible parametrizations of $\varphi_t^{-1}(G \cap \partial B(0,2)) \subseteq G_t$ will be parametrized at constant speed. Indeed,

$$s = d_t(|\gamma_t(s)|) = g_t(|\gamma_t(s)|) d(\varphi_t(\gamma_t(s))) = 2a|\gamma_t(s)|$$

for some constant $a = d(\varphi_t(\gamma_t(s)))$. In particular $G_1 \cap B(0,2)$ will consist of straight lines, parametrized in a linear fashion.

If $G \in \Phi(U)$ and $e : \mathbb{R}^N \to U$ is an embedding such that $e^*(G)$ is in collapsible position, we can define a path in $\Phi(e(\mathbb{R}^N))$ by $t \mapsto (e^{-1})^* \circ \Upsilon_t \circ e^*(G)$. This path is constant on $\Phi(e(\mathbb{R}^N - B(0,3)))$ so by the sheaf property (proposition 2.10) it glues with the constant path $t \mapsto G|(U - e(B(0,3)))$ to a path $t \mapsto \Upsilon_t^e(G) \in \Phi(U)$. This defines a continuous function $\Upsilon^e : [0,1] \times C(e) \to \Phi(U)$, where $C(e) \subseteq \Phi(U)$ is the open subset consisting of G for which $e^*(G)$ is in collapsible position.

Lemma 3.10. *For any $(G,f) \in \Phi(M;R)$ and any $v \in \mathscr{V}(R) \cap \mathrm{int}M$, let $T_v = f^{-1}(v)$. There exists an embedding $e = e_v : \mathbb{R}^N \to \mathrm{int}M$ such that $(e^{-1}(G), e^{-1}(T_v))$ is collapsible. If $W \subseteq U$ is a neighborhood of T_v then e_v can be chosen to have image in W. In particular we can choose the embeddings $e_v, v \in \mathscr{V}(R) \cap M$ to have disjoint images.*

Proof. Embed small disks around each vertex of T_v, and do connected sum along a small tubular neighborhood of each edge of T_v. □

If the embeddings in the above lemma have disjoint images, we get an embedding $e : (\mathscr{V}(R) \cap \mathrm{int}M) \times \mathbb{R}^N \to \mathrm{int}M$. We will say that e and (G, f) are *compatible* if they satisfy the conclusion of the lemma: $(e_v^{-1}(G), e_v^{-1}(T_v))$ is collapsible for all $v \in (\mathscr{V}(R) \cap \mathrm{int}M)$, where $e_v = e(v, -) : \mathbb{R}^N \to \mathrm{int}M$. Thus the lemma says that for any $(G, f) \in \Phi(M; R)$ we can find arbitrarily small compatible embeddings e.

From a compatible embedding $e : (\mathscr{V}(R) \cap \mathrm{int}M) \times \mathbb{R}^N \to \mathrm{int}M$ we construct a path $t \mapsto \Upsilon_t^e(G) \in \Phi(U)$ as above, i.e. by gluing the path

$$t \mapsto \prod_v (e_v^{-1})^* \circ \Upsilon_t \circ e_v^*(G) \in \Phi(\coprod_v e_v(\mathbb{R}^N))$$

with the constant path

$$t \mapsto G|(U - \coprod_v e_v(\mathbb{R}^N)).$$

If e is compatible with (G, f), the path $t \mapsto \Upsilon_t^e(G) \in \Phi(U)$ has a unique lift to a path $[0, 1] \to \Phi(M; R)$ which starts at (G, f). We will use the same notation $t \mapsto \Upsilon_t^e(G, f)$ for the lifted path $[0, 1] \to \Phi(M; R)$. We point out that $\Upsilon_t^e(G, f) \in \mathrm{Emb}_R(M)$ if $(G, f) \in \mathrm{Emb}_R(M)$ and that $\Upsilon_1^e(G, f) \in \mathrm{Emb}_R(M)$ for all $(G, f) \in \Phi(M; R)$. If we let $C(e) \subseteq \Phi(M; R)$ be the set of (G, f) which are compatible with e, we have constructed a *homotopy* between the inclusion $C(e) \subseteq \Phi(M; R)$ and the map $\Upsilon_1^e : C(e) \to \mathrm{Emb}_R(M)$.

We can now prove that $\pi_0(\Phi(M; R), \mathrm{Emb}_R(M)) = 0$. Namely, let $(G, f) \in \Phi(M; R)$ and use lemma 3.10 to choose a compatible $e : (\mathscr{V}(R) \cap \mathrm{int}M) \times \mathbb{R}^N \to \mathrm{int}M$. Then $t \mapsto \Upsilon_t^e(G, f)$ is a path to a point in $\mathrm{Emb}_R(M)$ as required.

To prove that the higher relative homotopy groups vanish, we need to carry out the above collapsing in families, i.e. when parametrized by a map $X \to \Phi(M; R)$. Unfortunately it does not seem easy to prove a parametrized version of lemma 3.10. Instead we use collapsing along multiple $e : \mathbb{R}^N \to U$ at once.

Definition 3.11. Let Q be the set of all embeddings $e : (\mathscr{V}(R) \cap M) \times \mathbb{R}^N \to M$. Write $e < e'$ if $e(\{v\} \times \mathbb{R}^N) \subseteq e'(\{v\} \times B(0, 1))$ for all $v \in \mathscr{V}(R)$. This makes Q into a poset. We give Q the discrete topology. Let $P \subseteq Q \times \Phi(M; R)$ be the subspace consisting of $(e, (G, f))$ with $(G, f) \in C(e)$. P is topologized in the product topology and ordered in the product ordering, where $\Phi(M; R)$ has the trivial order.

Lemma 3.12. *The projection $p : BP \to \Phi(M; R)$ is a weak equivalence. The restriction to $p^{-1}(\mathrm{Emb}_R(M)) \to \mathrm{Emb}_R(M)$ is also a weak equivalence.*

The proof is based on lemma 3.4. First recall that any poset D can be considered as a category. It is easy to see that BD is contractible when it has a subset $C \subseteq D$ which in the induced ordering is totally ordered and *cofinal*, i.e. if for every $d, d' \in D$ there is $c \in C$ with $d \leq c$ and $d' \leq c$. Indeed, any finite subcomplex of $\|N_\bullet D\|$ will be contained in the star of some vertex $c \in \|N_\bullet C\| \subseteq \|N_\bullet D\|$.

Proof of lemma 3.12. p is induced by the projection $\pi : P \to \Phi(M; R)$ which is etale (because $N_k P \subseteq N_k Q \times \Phi(M; R)$ is open and $N_k Q$ is discrete), so by lemma 3.4 it suffices to prove that $B(\pi^{-1}(G, f))$ is contractible for all (G, f). For any (G, f) we can choose, by lemma 3.10, a sequence $e_n : (\mathscr{V}(R) \cap \mathrm{int} M) \times \mathbb{R}^N \to \mathrm{int} M$, $n \in \mathbb{N}$ of embeddings compatible with (G, f), such that $e_1 > e_2 > \dots$, and with $e_n(\{v\} \times \mathbb{R}^N)$ contained in the $(1/n)$-neighborhood of T_v. This totally ordered subset of $\pi^{-1}((G, f))$ is cofinal. The second part is proved the same way. □

Definition 3.13. For $t = (t_0, \dots, t_k) \in [0,1]^{k+1}$ and $\chi = (e_0 < \dots < e_k, (G, f)) \in N_k(P)$, let

$$\Upsilon((t_0, \dots, t_k), \chi) = \Upsilon_{t_k}^{e_k} \circ \dots \circ \Upsilon_{t_0}^{e_0}(G, f).$$

This defines a continuous map $\Upsilon : [0,1]^{k+1} \times N_k(P) \to \Phi(M; R)$.

Proposition 3.14. *The inclusion* $\mathrm{Emb}_R(M) \to \Phi(M; R)$ *is a weak equivalence.*

Proof. Let $m : \Delta^k \to [0,1]^{k+1}$ be defined by

$$m(t_0, \dots, t_k) = (t_0, \dots, t_k)/\max(t_0, \dots, t_k).$$

Then the maps $h : [0,1] \times \Delta^k \times N_k(P) \to \Phi(M; R)$ defined by

$$h(\tau, t, \chi) = \Upsilon(\tau m(t), \chi)$$

glue together to a map $h : [0,1] \times BP \to \Phi(M; R)$ which is a homotopy between the projection map $p : BP \to \Phi(M; R)$ and the map

$$q = h(1, -) : BP \to \mathrm{Emb}_R(M) \subseteq \Phi(M; R).$$

This produces a null homotopy of the map of pairs

$$p : (BP, p^{-1}(\mathrm{Emb}_R(M))) \to (\Phi(M; R), \mathrm{Emb}_R(M)),$$

which together with lemma 3.12 proves that $\pi_*(\Phi(M; R), \mathrm{Emb}_R(M)) = 0$. □

Proposition 3.15. *The space* $\mathrm{Emb}_R(M)$ *is* $(N-4)$*-connected when* $\mathrm{int} M$ *is* $(N-3)$*-connected.*

Proof sketch. Thinking about $\mathrm{Emb}_R(M)$ as a space of embeddings $j : R \to U$ this is mostly standard, although the presence of vertices deserves some comment.

Firstly, we can fix j on $\mathscr{V}(R) \cap \mathrm{int}M$, since this changes homotopy groups only in degrees above $(N-4)$. Secondly, the proof of proposition 3.14 shows that we can assume there is a ball $B = B(v, \varepsilon_v)$ around each vertex $v \in \mathscr{V}(R) \cap \mathrm{int}M$ such that j is linear on $j^{-1}(B)$ (the point is that by construction, this holds after applying Υ_1^e). Thirdly, we can fix j on $j^{-1}B$, since this changes homotopy groups only in degrees above $(N-3)$. Then we are reduced to considering embeddings of a disjoint union of intervals into the manifold $\mathrm{int}M - \cup_v \mathrm{int}B(v, \varepsilon_v)$, and these form an $(N-4)$-connected space when $\mathrm{int}M$ is $(N-3)$-connected (an easy consequence of transversality). □

Proof of theorem 3.2. We have the maps

$$B\mathcal{G}_S \leftarrow \operatorname*{hocolim}_{R \in \mathcal{G}_S} \Phi(M; R) \to \Phi^S(M).$$

The map pointing to the right is a weak equivalence by proposition 3.6. The map pointing to the left is obtained by taking hocolim of the collapse map

$$\Phi(M; R) \to \textit{point}$$

which is $(N-3)$-connected by propositions 3.14 and 3.15. □

3.3. **Abstract graphs.** The goal in this section is to determine the homotopy type of the space $B\mathcal{G}_S$. Although the objects of $\mathcal{G}_S$ are embedded graphs, the embeddings play no role in the morphisms, and the category $\mathcal{G}_S$ is equivalent to a combinatorially defined category of *abstract* graphs. We recall the definition of abstract graphs, cf. [Ger84].

Definition 3.16. (i) A finite *abstract graph* is a finite set G with an involution $\sigma : G \to G$ and a retraction $t : G \to G^\sigma$ onto the fixed point set of σ.

(ii) The *vertices* of G is the set G^σ of fixed points of σ, and the complement $G - G^\sigma$ is the set of *half-edges*. The *valence* of a vertex $x \in G^\sigma$ is the number $v(x) = |t^{-1}(x)| - 1$. In this paper, all graphs are assumed not to have vertices of valence 0 and 2.

(iii) A *leaf* of G is a valence 1 vertex. A *leaf labelling* of G is an identification of the set of leaves of G with $\{1, \dots, s\}$.

(iv) A *cellular map* $G \to G'$ between two abstract graphs is a set map preserving σ and t.

(v) A cellular map $G \to G'$ is a *graph epimorphism* if the inverse image of each half-edge of G' is a single half-edge of G, and the

inverse image of a vertex of G' is a *tree* (i.e. contractible graph), not containing any leaves. If the leaves of G and G' are labelled, we require the map to preserve the labelling.

(vi) For $s \geq 0$, let $\mathcal{G}_s$ denote the category whose objects are finite abstract graphs with leaves labelled by $\{1, \dots, s\}$ and whose morphisms are graph epimorphisms.

A cellular map is a graph epimorphism if and only if it can be written as a composition of *isomorphisms* and *elementary collapses*, i.e. maps which collapse a single non-loop, non-leaf edge to a point.

Lemma 3.17. *Let $N \geq 3$ and let $M \subseteq U$ be compact with* $\mathrm{int}M$ *connected. Let $S \in \Phi(U - \mathrm{int}M)$ be a germ of a graph with $s \geq 0$ ends in* $\mathrm{int}M$ *(i.e. s is the cardinality of the inverse limit of $\pi_0(S \cap (\mathrm{int}M - K))$ over larger and larger compact sets $K \subseteq \mathrm{int}M$). Then we have an equivalence of categories*

$$\mathcal{G}_S \simeq \mathcal{G}_s. \qquad \square$$

It remains to determine the homotopy type of the space $B\mathcal{G}_s$. A finite abstract graph G has a *realization*

$$|G| = \big(G^\sigma \amalg ((G - G^\sigma) \times [-1, 1])\big)/\sim,$$

where $\sim$ is the equivalence relation generated by $(x, r) \sim (\sigma x, -r)$ and $(x, 1) \sim t(x) \in G^\sigma$ for $x \in G - G^\sigma$, $r \in [-1, 1]$. Let $\partial|G| \subseteq |G|$ be the set of leaves (valence 1 vertices). Let $\mathrm{Aut}(G)$ denote the group of homotopy classes of homotopy equivalences $|G| \to |G|$ restricting to the identity on $\partial|G|$. Recall from section 1.2 that $A_n^s = \mathrm{Aut}(G_n^s)$, where G_n^s is a graph with first Betti number n and s leaves. In particular $A_n^0 = \mathrm{Out}(F_n)$ and $A_n^1 = \mathrm{Aut}(F_n)$.

$B\mathcal{G}_0$ and $B\mathcal{G}_1$ are directly related to automorphisms of free groups, via Culler-Vogtmann's *outer space* [CV86]. As mentioned in section 1.2, outer space is a contractible space with an action of $\mathrm{Out}(F_n)$. Culler-Vogtmann also define a certain subspace called the *spine of outer space*, which is an equivariant deformation retract. For a finite abstract graph G_0, the spine of outer space $X(G_0)$ has one simplex for each isomorphism class of pairs (G, h), where G is a finite abstract graph and $h : |G| \to |G_0|$ is a homotopy class of a homotopy equivalence. (G, h) is a face of (G', h') when there exists a graph epimorphism $\varphi : G' \to G$ such that $h \circ |\varphi| \simeq h'$. Culler-Vogtmann prove that $X(G_0)$ is contractible. (They state this only for connected G_0; the general statement follows from the homeomorphism $X(G_0 \amalg G_1) \cong X(G_0) \times X(G_1)$.)

Proposition 3.18. *There is a homotopy equivalence*

$$B\mathcal{G}_s \simeq \coprod_G B\mathrm{Aut}(G), \tag{3.2}$$

where the disjoint union is over finite graphs G with s leaves, one of each homotopy type.

The right hand side of the homotopy equivalence (3.2) can conveniently be reformulated in terms of a category $\mathcal{G}_s^{\simeq}$. The objects of $\mathcal{G}_s^{\simeq}$ are the objects of $\mathcal{G}_s$, but morphisms $G \to G'$ in $\mathcal{G}_s^{\simeq}$ are homotopy classes of homotopy equivalences $(|G|, \partial|G|) \to (|G'|, \partial|G'|)$, compatible with the labellings. We have inclusion functors

$$\mathcal{G}_s \xrightarrow{f} \mathcal{G}_s^{\simeq} \xleftarrow{g} \coprod_G \mathrm{Aut}(G)$$

where f is the identity on the set objects and takes geometric realization of morphisms, and g is the inclusion of a skeletal subcategory. Consequently g is an equivalence of categories, and the statement of proposition 3.18 is equivalent to f inducing a homotopy equivalence $Bf : B\mathcal{G}_s \to B\mathcal{G}_s^{\simeq}$.

Proof sketch. We first consider the case $s = 0$, following [Igu02, Theorem 8.1.21].

For a fixed object $G_0 \in \mathcal{G}^{\simeq}$, we consider the over category $(\mathcal{G} \downarrow G_0)$. Its objects are pairs (G, h) consisting of an object $G \in \mathrm{ob}(\mathcal{G})$ and a homotopy class of a homotopy equivalence $h : |G| \to |G_0|$. Its morphisms $(G, h) \to (G', h')$ are graph epimorphisms $\varphi : G \to G'$ with $h' \circ |\varphi| \simeq h$. It is equivalent to the opposite of the poset of simplices in the spine of outer space, and hence contractible. Then the claim follows from Quillen's "theorem A" ([Qui73]).

We proceed by induction in s. Recall that $\mathrm{Aut}(G) = \pi_0 h\mathrm{Aut}(G)$, where $h\mathrm{Aut}(G)$ is the topological monoid of self-homotopy equivalences of $|G|$ restricting to the identity on the boundary. Every connected component of $h\mathrm{Aut}(G)$ is contractible and we have $Bh\mathrm{Aut}(G) \simeq B\mathrm{Aut}(G)$. The monoid $h\mathrm{Aut}(G)$ acts on $|G|$, and the Borel construction is

$$Eh\mathrm{Aut}(G) \times_{h\mathrm{Aut}(G)} |G| \simeq \coprod_p Bh\mathrm{Aut}(G'),$$

where G' is obtained by attaching an extra leaf to G at a point p. The disjoint union is over $p \in |G| - \partial|G|$, one in each $h\mathrm{Aut}(G)$-orbit. It follows that the map

$$B\mathcal{G}_{s+1}^{\simeq} \to B\mathcal{G}_s^{\simeq},$$

induced by forgetting the leaf labelled $s+1$, has homotopy fiber $|G|$ over the point $G \in B\mathcal{G}_s^{\simeq}$.

Let $\Gamma : \mathcal{G}_s \to \mathrm{CAT}$ be the functor which to $G \in \mathcal{G}_s$ associates the poset of simplices of G which are not valence 1 vertices, ordered by reverse inclusion. Recall from e.g. [Tho79] that to such a functor there is a associated category $(\mathcal{G}_s \wr \Gamma)$. An object of the category $(\mathcal{G}_s \wr \Gamma)$ is a pair (G, σ), with $G \in \mathcal{G}_s$ and $\sigma \in \Gamma(G)$ and a morphism $(G, \sigma) \to (G', \sigma')$ is a pair (φ, ψ) with $\varphi : G \to G'$ and $\psi : \Gamma(\varphi)(\sigma) \to \sigma'$. There is a functor

$$(\mathcal{G}_s \wr \Gamma) \to \mathcal{G}_{n+1}$$

which maps (G, σ) to the graph obtained by attaching a leaf labeled $s+1$ to G at the barycenter of σ. This is an equivalence of categories, and it follows (by [Tho79]) that the homotopy fiber of the projection $B\mathcal{G}_{s+1} \to B\mathcal{G}_s$ over the point $G \in B\mathcal{G}_s$ is $B(\Gamma(G)) \cong |G|$.

Therefore the diagram

$$\begin{tikzcd}
B\mathcal{G}_{s+1} \arrow[r] \arrow[d] & B\mathcal{G}_{s+1}^{\simeq} \arrow[d] \\
B\mathcal{G}_s \arrow[r] & B\mathcal{G}_s^{\simeq}
\end{tikzcd}$$

is homotopy cartesian. This proves the induction step. □

Summarizing theorem 3.2, lemma 3.17, and proposition 3.18 we get

Theorem 3.19. *Let $N \geq 3$, let $U \subseteq \mathbb{R}^N$ be open, and let $M \subseteq U$ be compact with* $\mathrm{int}M$ *$(N-3)$-connected. Let $S \in \Phi(U - \mathrm{int}M)$ be a germ of a graph with s ends in* $\mathrm{int}M$. *Then we have an $(N-3)$-connected map*

$$\Phi^S(M) \to \coprod_G B\mathrm{Aut}(G), \tag{3.3}$$

where the disjoint union is over finite graphs G with s leaves, one of each homotopy type.

3.4. $B\mathrm{Out}(F_n)$ **and the graph spectrum.** We are now ready to begin the proof outlined in subsection 1.2. The first goal is to define the maps (1.2) and (1.3). The space B_N in the following definition is the domain of the map (1.2).

Definition 3.20. Let $I = [-1, 1]$. Let $B_N \subseteq \Phi(\mathbb{R}^N)$ be the subset

$$B_N = \Phi^{[\emptyset]}(I^N),$$

i.e. the set of graphs contained in $\mathrm{int}(I^N)$.

The homotopy type of the space B_N is determined by theorem 3.19.

Proposition 3.21. *There is an $(N-3)$-connected map*

$$B_N \to \coprod_G B\mathrm{Aut}(G),$$

where the disjoint union is over graphs G without leaves, one of each homotopy type. Consequently we have a weak equivalence

$$\{G \in B_\infty \mid \textit{there exists a homotopy equivalence } G \simeq \vee^n S^1\} \simeq B\mathrm{Out}(F_n). \qquad \square$$

Approximating $B\mathrm{Out}(F_n)$ by the space consisting of $G \in B_N$ for which there exists a homotopy equivalence $G \simeq \vee^n S^1$ is analogous to the approximation

$$B\mathrm{Diff}(M) \sim \mathrm{Emb}(M, \mathrm{int}(I^N))/\mathrm{Diff}(M)$$

for a smooth manifold M. The right hand side is the space of submanifolds $Q \subseteq \mathrm{int}(I^N)$ for which there exists a diffeomorphism $Q \cong M$.

The empty set $\emptyset \subseteq \mathbb{R}^N$ is a graph, and we consider it the basepoint of $\Phi(\mathbb{R}^N)$.

Definition 3.22. Let $\varepsilon_N : S^1 \wedge \Phi(\mathbb{R}^N) \to \Phi(\mathbb{R}^{N+1})$ be the map induced by the map $\mathbb{R} \times \Phi(\mathbb{R}^N) \to \Phi(\mathbb{R}^{N+1})$ given by $(t, G) \mapsto \{-t\} \times G$.

Lemma 3.23. *ε_N is well defined and continuous.*

Proof. For any compact subset $K \subseteq \mathbb{R}^N$ with $K \subseteq cD^{N+1}$ we will have $(\{t\} \times G) \cap K = \emptyset \in \Phi(\mathbb{R}^{N+1})$ as long as $|t| > c$ or $G \cap cD^N = \emptyset$. This proves that ε_N is continuous at the basepoint. Continuity on $\mathbb{R} \times \Phi(\mathbb{R}^N)$ follows from proposition 2.11. $\square$

Definition 3.24. Let $\mathbf{\Phi}$ be the spectrum with Nth space $\Phi(\mathbb{R}^N)$ and structure maps ε_N. This is the *graph spectrum.*

We will not use any theory about spectra. In fact we will always work with the corresponding infinite loop space $\Omega^\infty \mathbf{\Phi}$ defined as

$$\Omega^\infty \mathbf{\Phi} = \operatorname*{colim}_{N \to \infty} \Omega^N \Phi(\mathbb{R}^N)$$

where the map $\Omega^N \Phi(\mathbb{R}^N) \to \Omega^{N+1} \Phi(\mathbb{R}^{N+1})$ is the N-fold loop of the adjoint of ε_N.

$\mathbf{\Phi}$ is the analogue for graphs of the spectrum $MTO(d)$ for d-manifolds in the paper [GMTW06]. The analogy is clarified in chapter 6, especially proposition 6.2. $MTO(d)$ is the Thom spectrum of the universal stable normal bundle for d-manifold bundles, $-U_d \to BO(d)$. Thus $\mathbf{\Phi}$ is a kind of "Thom spectrum of the universal stable normal bundle for

graph bundles." Remark 5.13 explains in what sense $\mathbf{\Phi}$ is the Thom spectrum of a "generalized stable spherical fibration". In this subsection we will define a map which, alluding to a similar analogy, we could call the *parametrized Pontryagin-Thom collapse map for graphs*

$$B\mathrm{Out}(F_n) \to \Omega^\infty \mathbf{\Phi}. \tag{3.4}$$

Given $G \in B_N$ and $v \in \mathbb{R}^N$ we can translate G by v and get an element

$$\tau_N(G)(v) = G - v \in \Phi(\mathbb{R}^N).$$

We have $\tau_N(G)(v) \to \emptyset$ if $|v| \to \infty$, so τ_N extends uniquely to a continuous map

$$(B_N) \wedge S^N \xrightarrow{\tau_N} \Phi(\mathbb{R}^N). \tag{3.5}$$

Definition 3.25. Let $\tau_N : B_N \to \Omega^N \Phi(\mathbb{R}^N)$ be the adjoint of the map (3.5).

In the following diagram, the left vertical map $B_N \to B_{N+1}$ is the inclusion $G \mapsto \{0\} \times G$.

$$\begin{tikzcd} B_N \arrow[r, "\tau_N"] \arrow[d] & \Omega^N \Phi(\mathbb{R}^N) \arrow[d, "\varepsilon_N"] \\ B_{N+1} \arrow[r, "\tau_{N+1}"] & \Omega^{N+1}\Phi(\mathbb{R}^{N+1}). \end{tikzcd}$$

The diagram is commutative, and we get an induced map

$$\tau_\infty : B_\infty \to \Omega^\infty \mathbf{\Phi}. \tag{3.6}$$

By theorem 4.3, $B\mathrm{Out}(F_n)$ is a connected component of B_∞, and we define the map (3.4) as the restriction of (3.6).

The map τ_N is homotopic to a map $\tilde{\tau}_N$ defined in a different way. This construction will be used in section 4.1.2, but is not logically necessary for the proof of theorem 1.1. $\tilde{\tau}_N$ is similar to the "scanning" map of [Seg79], and is defined as follows. Choose a map

$$e : \mathrm{int}(I^N) \times \mathbb{R}^N = T\mathrm{int}(I^N) \to \mathrm{int}(I^N)$$

such that for each $p \in \mathrm{int}(I^N)$, the induced map

$$e_p : T_p\mathrm{int}(I^N) \to \mathrm{int}(I^N).$$

is an embedding with $e_p(0) = p$ and $De_p(0) = \mathrm{id}$. We can arrange that the radius of $e_p(T_p\mathrm{int}(I^N))$ is smaller than the distance $\mathrm{dist}(p, \partial I^N)$. To a graph $G \in B_N$ and a point $p \in \mathrm{int}(I^N)$ we associate

$$\tilde{\tau}(G)(p) = (e_p)^*(G) \in \Phi(T_p\mathbb{R}^N) = \Phi(\mathbb{R}^N).$$

By proposition 2.11, the action of $\mathrm{Diff}(\mathbb{R}^k)$ on $\Phi(\mathbb{R}^k)$ is continuous. Therefore we can apply Φ fiberwise to vector bundles: If $V \to X$ is a vector bundle, then there is a fiber bundle $\Phi^{\mathrm{fib}}(V)$ whose fiber over x is $\Phi(V_x)$. We will have $\tilde{\tau}_N(G)(p) = \emptyset$ for all p outside some compact subset of $\mathrm{int}(I^N)$. $\tilde{\tau}_N(G)(p)$ is continuous as a function of p, and can be interpreted as a *compactly supported section* over $\mathrm{int}(I^N)$ of the fiber bundle $\Phi^{\mathrm{fib}}(T\mathbb{R}^N)$. We define $\tilde{\tau}_N(G)(p) = \emptyset$ for $p \in \partial I^N$ and get a section

$$\tilde{\tau}_N(G) \in \Gamma((I^N, \partial I^N), \Phi^{\mathrm{fib}}(T\mathbb{R}^N)) \cong \Omega^N \Phi(\mathbb{R}^N).$$

$\tilde{\tau}_N(G)$ depends continuously on G so we get a continuous map

$$\tilde{\tau}_N : B_N \to \Omega^N \Phi(\mathbb{R}^N), \tag{3.7}$$

which is easily seen to be homotopic to the map τ_N of definition 3.25.

4. The graph cobordism category

As explained in the introduction, composing (3.4) with the maps $B\mathrm{Aut}(F_n) \to B\mathrm{Aut}(F_{n+1}) \to B\mathrm{Out}(F_{n+1})$ gives a map

$$\coprod_{n \geq 0} B\mathrm{Aut}(F_n) \xrightarrow{\tau} \Omega^\infty \mathbf{\Phi}. \tag{4.1}$$

We will prove the following.

Theorem 4.1. *τ induces a homology equivalence*

$$\mathbb{Z} \times B\mathrm{Aut}_\infty \to \Omega^\infty \mathbf{\Phi}.$$

$\coprod B\mathrm{Aut}(F_n)$ is a topological *monoid* whose group completion is $\mathbb{Z} \times B\mathrm{Aut}_\infty^+$. It turns out to be fruitful to enlarge it to a topological *category*, with more than one object. Namely we will define a "graph cobordism category" $\mathcal{C}_N$ whose morphisms are graphs in $\mathbb{R}^N$.

Definition 4.2. For $\varepsilon > 0$, let $\mathrm{ob}(\mathcal{C}_N^\varepsilon)$ be the set

$$\{(a, A, \lambda) \mid a \in \mathbb{R},\ A \subseteq \mathrm{int}(I^{N-1}) \text{ finite},\ \lambda \in (-1+\varepsilon, 1-\varepsilon)^A\}.$$

For an object $c = (a, A, \lambda)$, let $U_a^\varepsilon = (a-\varepsilon, a+\varepsilon) \times \mathbb{R}^{N-1}$ and let

$$S_c^\varepsilon = (a-\varepsilon, a+\varepsilon) \times A.$$

Equipped with the map $l_c : S_c \to [0,1)$ given by

$$l_c(a+t, x) = (t + \lambda(x))^2$$

for $|t| < \varepsilon$, this defines an element $(S_c^\varepsilon, l_c) \in \Phi(U_a^\varepsilon)$. For two objects $c_0 = (a_0, A_0, \lambda_0)$ and $c_1 = (a_1, A_1, \lambda_1)$ with $0 < 2\varepsilon < a_1 - a_0$, let $\mathcal{C}_N^\varepsilon(c_0, c_1)$ be the set consisting of $(G, l) \in \Phi((a_0 - \varepsilon, a_1 + \varepsilon) \times \mathbb{R}^{N-1})$ satisfying $(G, l)|U_{a_\nu}^\varepsilon = (S_{c_\nu}^\varepsilon, l_{c_\nu})$ for $\nu = 0, 1$. If $c_2 = (a_2, A_2, \lambda_2)$ is

a third object and $(G', l') \in \mathcal{C}_N^\varepsilon(c_1, c_2)$, let $(G, l) \circ (G', l') = (G'', l'')$, where

$$G'' = G \cup G'$$

and $l'' : G'' \to [0, 1]$ agrees with l on G and with l' on G'. This defines $\mathcal{C}_N^\varepsilon$ as a category of sets. Topologize the total set of morphisms as a subspace of

$$\coprod_{a_0, a_1} \Phi((a_0 - \varepsilon, a_1 + \varepsilon) \times \mathbb{R}^{N-1}),$$

where the coproduct is over $a_0, a_1 \in \mathbb{R}$ with either $a_0 = a_1$ (the identities) or $0 < 2\varepsilon < a_1 - a_0$. We have inclusions $\mathcal{C}_N^\varepsilon \to \mathcal{C}_N^{\varepsilon'}$ when $\varepsilon' < \varepsilon$, and we let

$$\mathcal{C}_N = \operatorname*{colim}_{\varepsilon \to 0} \mathcal{C}_N^\varepsilon.$$

The following theorem determines the homotopy type of the space of morphisms between two fixed objects in $\mathcal{C}_N$. It is a consequence of theorem 3.2, lemma 3.17, and proposition 3.18.

Theorem 4.3. *Let $c_0 = (a_0, A_0, \lambda_0)$ and $c_1 = (a_1, A_1, \lambda_1)$ be objects of $\mathcal{C}_N$ with $a_0 < a_1$. There is an $(N-3)$-connected map*

$$\mathcal{C}_N(c_0, c_1) \to \coprod_G B\mathrm{Aut}(G),$$

where the disjoint union is over finite graphs G with $s = |A_0| + |A_1|$ leaves, one of each homotopy type. Consequently

$$\{G \in \mathcal{C}_\infty(c_0, c_1) \mid G \text{ is connected}\} \simeq \coprod_{n \geq 0} BA_n^s. \tag{4.2}$$

($n = 0$ should be excluded if $s = 1$ and $n = 0, 1$ should be excluded if $s = 0$.) □

For the proof of theorem 4.1 we need two more definitions.

Definition 4.4. Let $D_N \subseteq \Phi(\mathbb{R}^N)$ denote the subspace

$$\{G \in \Phi(\mathbb{R}^N) \mid G \subseteq \mathbb{R} \times \mathrm{int}(I^{N-1})\}.$$

Definition 4.5. The *positive boundary* subcategory $\mathcal{C}_N^\partial \subseteq \mathcal{C}_N$ is the subcategory with the same space of objects, but whose space of morphisms from $c_0 = (a_0, A_0, \lambda_0)$ to $c_1 = (a_1, A_1, \lambda_1)$ is the subset

$$\{G \in \mathcal{C}_N(c_0, c_1) \mid A_1 \to \pi_0(G) \text{ surjective}\}.$$

Then theorem 4.1 is proved in the following four steps. Carrying them out occupies the remainder of this chapter.

- There is a homology equivalence
$$\mathbb{Z} \times B\mathrm{Aut}_\infty \to \Omega B\mathcal{C}^\partial_\infty. \tag{4.3}$$
- There is a weak equivalence
$$B\mathcal{C}_N \simeq D_N. \tag{4.4}$$
- There is a weak equivalence
$$D_N \xrightarrow{\simeq} \Omega^{N-1}\Phi(\mathbb{R}^N). \tag{4.5}$$
- The inclusion induces a weak equivalence
$$B\mathcal{C}^\partial_\infty \xrightarrow{\simeq} B\mathcal{C}_\infty. \tag{4.6}$$

Then theorem 4.1 follows by looping (4.4), (4.5), and (4.6), taking the direct limit $N \to \infty$ in (4.4) and (4.5), and composing.

4.1. **Poset model of the graph cobordism category.** We will use $\mathbb{R}^\delta$ to denote the set $\mathbb{R}$ of real numbers, equipped with the *discrete* topology.

Definition 4.6. (i) Let $D^\pitchfork_N \subseteq \mathbb{R}^\delta \times D_N$ be the space of pairs $(a, (G, l))$ satisfying
$$G \pitchfork \{a\} \times \mathbb{R}^{N-1}. \tag{4.7}$$
This is a poset, with ordering defined by $(a_0, G) \leq (a_1, G')$ if and only if $G = G'$ and $a_0 \leq a_1$.

(ii) For $\varepsilon > 0$, let $D^{\perp,\varepsilon}_N \subseteq D^\pitchfork_N$ be the subposet defined as follows. $(a, (G, l)) \in D^{\perp,\varepsilon}_N$ if there exists $c = (a, A, \lambda)$ as in definition 4.2 such that $(G, l)|U^\varepsilon_a = (S^\varepsilon_c, l_c)$.

(iii) Let $D^\perp_N$ be the colimit of $D^{\perp,\varepsilon}_N$ as $\varepsilon \to 0$.

There is an inclusion functor $i : D^\perp_N \to D^\pitchfork_N$, and a forgetful map $u : D^\pitchfork_N \to D_N$. There is also a functor $c : D^\perp_N \to \mathcal{C}_N$ defined as follows. Let $(x_0 < x_1) \in N_1 D^{\perp,\varepsilon}_N$ with $x_0 = (a_0, G)$, $x_1 = (a_1, G)$. Then let
$$c(x_0 < x_1) = G|(a_0 - \varepsilon, a_1 + \varepsilon) \times \mathbb{R}^{N-1}.$$
This defines a functor $D^{\perp,\varepsilon}_N \to \mathcal{C}^\varepsilon_N$, and $c : D^\perp_N \to \mathcal{C}_N$ is defined by taking the colimit.

The following lemma implies proposition 4.8.

Lemma 4.7. *The induced maps*
$$Bi : BD^\perp_N \to BD^\pitchfork_N \tag{4.8}$$
$$Bu : BD^\pitchfork_N \to D_N \tag{4.9}$$
$$Bc : BD^\perp_N \to B\mathcal{C}_N \tag{4.10}$$

are all weak equivalences.

Proof. In fact (4.8) and (4.10) are both induced by degreewise weak homotopy equivalences on simplicial nerves. For (4.8) this is obvious—straighten the morphisms near their ends.

For (4.9), notice that all maps

$$N_k u : N_k D_N^{\pitchfork} \to D_N$$

are etale, and that for $G \in D_N$, the inverse image $u^{-1}(G)$ is the set

$$\{a \in \mathbb{R} \mid G \pitchfork \{a\} \times \mathbb{R}^{N-1}\}$$

which has the discrete topology, is non-empty, and totally ordered. It follows that $B(u^{-1}(G))$ is contractible, and the claim follows from lemma 3.4.

For (4.10), suppose P is a sphere and $f : P \to N_k\mathcal{C}_N$ a continuous map. By compactness, f maps into $N_k\mathcal{C}_N^{\varepsilon}$ for some $\varepsilon > 0$ so all graphs in the image of f are elements of $\Phi((a_0 - \varepsilon, a_k + \varepsilon) \times \mathbb{R}^N)$. Choose a diffeomorphism from $(a_0 - \varepsilon, a_k + \varepsilon)$ to $\mathbb{R}$ which is the identity on $(a_0 - \varepsilon/2, a_k + \varepsilon/2)$ and use that to lift f to $P \to N_k D_N^{\perp}$. We have constructed an inverse to $\pi_*(N_k c)$. □

We have proved the following result.

Proposition 4.8. *There is a weak equivalence*

$$B\mathcal{C}_N \simeq D_N. \qquad \square$$

A variation of the proof of proposition 4.8 given above will prove the following result. The details are given below.

Proposition 4.9. *There is a weak equivalence*

$$D_N \xrightarrow{\simeq} \Omega^{N-1}\Phi(\mathbb{R}^N).$$

The map $D_N \to \Omega^{N-1}\Phi(\mathbb{R}^N)$ is similar to the map τ_N in definition 3.25. First let $\mathbb{R}^{N-1} \times D_N \to \Phi(\mathbb{R}^N)$ be given by the formula

$$(v, G) \mapsto G - (0, v). \tag{4.11}$$

This extends uniquely to a continuous map $S^{N-1} \wedge D_N \to \Phi(\mathbb{R}^N)$, and the adjoint of this map is the weak equivalence in proposition 4.9. This map is homotopic to a map

$$D_N \to \Gamma((\mathbb{R} \times I^{N-1}, \mathbb{R} \times \partial I^{N-1}), \Phi^{\mathrm{fib}}(T\mathbb{R}^N)) \simeq \Omega^{N-1}\Phi(\mathbb{R}^N) \tag{4.12}$$

defined by "scanning", just like the map τ_N in definition 3.25 is homotopic to the map $\tilde{\tau}_N$ in (3.7).

We give two proofs proposition 4.9. The first is a direct induction proof which is similar to the proofs of propositions 4.8 and 4.16 above.

The second uses Gromov's "flexible sheaves" [Gro86, section 2]. While this is somewhat heavy machinery, we believe it illuminates the relation between scanning maps and Pontryagin-Thom collapse maps nicely. For the second proof, the crucial properties of Φ are the continuity property expressed in proposition 2.11 and that Φ is "microflexible".

4.1.1. *First proof.* For $k = 0, 1, \dots, N$, let $D_{N,k} \subseteq \Phi(\mathbb{R}^N)$ be the subspace

$$D_{N,k} = \{G \in \Phi(\mathbb{R}^N) \mid G \subseteq \mathbb{R}^k \times \mathrm{int}(I^{N-k})\},$$

equipped with the subspace topology. In particular $D_{N,0} = B_N$, $D_{N,1} = D_N$ and $D_{N,N} = \Phi(\mathbb{R}^N)$. The map $\mathbb{R} \times D_{N,k-1} \to D_{N,k}$ given by

$$(t, G) \mapsto G - (0, t, 0)$$

extends uniquely to a continuous map $S^1 \wedge D_{N,k-1} \to D_{N,k}$ and we consider its adjoint

$$D_{N,k-1} \to \Omega D_{N,k}. \tag{4.13}$$

The composition of the maps (4.13) for $k = 2, \dots, N$, is the map $D_N \to \Omega^{N-1}\Phi(\mathbb{R}^N)$ of proposition 4.9.

Proposition 4.10. *The map* (4.13) *is a weak equivalence for* $k = 2, 3, \dots, N$.

Proposition 4.9 then follows from proposition 4.10 by induction. The proof of proposition 4.10 is given in the lemmas 4.12 and 4.13 below. The proofs of these lemmas are very similar to the proofs of the propositions 4.16 and 4.8, respectively.

Definition 4.11. Let $k \geq 2$.

(i) Let $D^{\pitchfork}_{N,k}$ be the space of triples $(G, a, p) \in D_{N,k} \times \mathbb{R}^\delta \times (\mathbb{R}^\delta)^{k-1}$ satisfying

$$\{p\} \times \{a\} \times \mathbb{R}^{N-k} \cap G = \emptyset. \tag{4.14}$$

Order $D^{\pitchfork}_{N,k}$ by declaring $(G, a, p) < (G', a', p')$ if and only if $G = G'$ and $a < a'$.

(ii) Let $D^{\perp}_{N,k} \subseteq D^{\pitchfork}_{N,k}$ be the subspace, and sub poset, consisting of triples satisfying the further condition

$$\mathbb{R}^{k-1} \times \{a\} \times \mathbb{R}^{N-k} \cap G = \emptyset. \tag{4.15}$$

(iii) Let $\mathcal{C}_{N,k}$ be the category whose space of objects is $\mathbb{R}^\delta \times (\mathbb{R}^\delta)^{k-1}$ and with morphism spaces given by

$$\mathcal{C}_{N,k}((a_0, p_0), (a_1, p_1)) = \{G \in \Phi(\mathbb{R}^N) \mid G \subseteq \mathbb{R}^{k-1} \times \mathrm{int}([a_0, a_1] \times I^{N-k})\},$$

when $a_0 \leq a_1$. Composition is union of subsets.

(iv) Let $i : D^{\perp}_{N,k} \to D^{\pitchfork}_{N,k}$ be the inclusion functor and let $u : D^{\pitchfork}_{N,k} \to D_{N,k}$ be the forgetful map. Let $c : D^{\perp}_{N,k} \to \mathcal{C}_{N,k}$ be the functor given on morphisms by

$$c((G, a_0, p_0) \leq (G, a_1, p_1)) = G \cap \big(\mathbb{R}^{k-1} \times [a_0, a_1] \times \mathbb{R}^{N-k}\big).$$

Lemma 4.12. *The maps*

$$B\mathcal{C}_{N,k} \xleftarrow{Bc} BD^{\perp}_{N,k} \xrightarrow{Bi} BD^{\pitchfork}_{N,k} \xrightarrow{Bu} D_{N,k}$$

are all weak equivalences.

Proof. This is very similar to the proof of lemma 4.7. We first consider Bu. For each l, $N_l u : N_l D^{\pitchfork}_{N,k} \to D_{N,k}$ is an etale map, and each fiber $u^{-1}(G)$ is a contractible poset (for each $a \in \mathbb{R}$, we can choose $p_a \in \mathbb{R}^{k-1}$ such that $(G, a, p_a) \in D^{\pitchfork}_{N,k}$. The set of all (G, a, p_a), $a \in \mathbb{R}$ forms a totally ordered cofinal subposet of $u^{-1}(G)$). The result now follows from lemma 3.4.

For Bi we claim that the inclusion $N_l D^{\perp}_{N,k} \to N_l D^{\pitchfork}_{N,k}$ is a deformation retract for each l. A non-degenerate element $\chi \in N_l D^{\pitchfork}_{N,k}$ is given by an element $G \in D_{N,k}$, real numbers $a_0 < \cdots < a_l$, and points $p_0, \dots, p_l \in \mathbb{R}^{k-1}$. We will define a path from χ to a point in $N_l D^{\perp}_{N,k}$ depending continuously on χ. In essence, the construction is a parametrized version of the path constructed in lemma 2.6.

For $r \in \mathbb{R}$, let $h_r : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ be the affine function given by

$$h_r(x) = x \prod_{i=0}^{l} (r - a_i)^2 + \sum_{i=0}^{l} p_i \prod_{j \neq i} \frac{r - a_j}{a_i - a_j}.$$

Then h_r is a diffeomorphism for $r \notin \{a_0, \dots, a_l\}$ and $h_{a_i}(x) = p_i$ for all x. For $t \in [0,1]$, let $\varphi_t : \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{N-k} \to \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{N-k}$ be the map given by

$$\varphi_t(x, r, y) = ((1-t)x + t h_r(x), r, y)$$

and let $G_t = (\varphi_t)^*(G)$. Then $G_0 = G$ and G_1 will satisfy (4.15) with respect to any $a = a_\nu$, $\nu \in \{0, 1, \dots, l\}$. This gives a continuous path in $N_l D^{\pitchfork}_{N,k}$,

$$t \mapsto \chi_t = (G_t, (a_0 < \cdots < a_l), (p_0, \dots, p_l)), \quad t \in [0,1],$$

starting at $\chi_0 = \chi$ and ending at $\chi_1 \in N_l D^{\perp}_{N,k}$.

Finally, Bc is a homotopy equivalence because $N_l c$ is a homotopy equivalence for all l. This is proved precisely as the analogous statement in lemma 4.7. $\square$

For the proof of lemma 4.13, recall that to a covariant functor $F : C \to \text{Spaces}$ there is an associated category $(C \wr F)$. When F is contravariant we let $(F \wr C) = (C^{\text{op}} \wr F)^{\text{op}}$.

Lemma 4.13. *Let $k \geq 2$.*

(i) $\mathcal{C}_{N,k}((a_0, p_0), (a_1, p_1)) \cong D_{N,k-1}$ *whenever* $a_0 < a_1$.

(ii) Composition with any morphism $G : (a_1, p_1) \to (a_2, p_2)$ *induces a homotopy equivalence*

$$\mathcal{C}_{N,k}((a_0, p_0), (a_1, p_1)) \xrightarrow{G\circ} \mathcal{C}_{N,k}((a_0, p_0), (a_2, p_2)).$$

Similarly for composing from the right.

(iii) There is a weak equivalence

$$D_{N,k} \simeq \Omega B\mathcal{C}_{N,k-1}.$$

Proof. *(i)* There is an obvious homeomorphism that stretches the interval $[a_0, a_1]$ to $[0, 1]$.

(ii) is clear in the case $G = \emptyset$. The space $D_N = D_{N,1} \simeq B\mathcal{C}_N$ is connected (given any two objects, there is a morphism between them). Similarly $D_{N,k} \simeq B\mathcal{C}_{N,k}$ is connected for $k \geq 2$. Therefore composition with any $G \in \mathcal{C}_{N,k}((a_1, p_1), (a_2, p_2)) \simeq D_{N,k-1}$ is homotopic to composition with $G = \emptyset$.

(iii). Consider for each $n \in \mathbb{Z}$ the object $(n, 0)$ in $\mathcal{C}_{N,k}$, and let $F_n : \mathcal{C}_{N,k} \to \text{Spaces}$ be the functor

$$F_n = \mathcal{C}_{N,k}(-, (n, 0)).$$

The morphism $\emptyset : (n, 0) \to (n + 1, 0)$ induces a natural transformation $F_n \to F_{n+1}$, and we let

$$F_\infty((a, p)) = \operatorname*{hocolim}_{n\to\infty} F_n((a, p)).$$

Then $\mathrm{id}_{(n,0)}$ is a final object of the category $(F_n \wr \mathcal{C}_{N,k})$, so $B(F_n \wr \mathcal{C}_{N,k})$ and $B(F_\infty \wr \mathcal{C}_{N,k})$ are contractible.

From *(i)* and *(ii)* we get that $F_\infty((a, p)) \simeq D_{N,k-1}$, and that any morphism in $\mathcal{C}_{N,k}$ induces a homotopy equivalence $F_\infty((a_1, p)) \to F_\infty((a_0, p))$. Then the simplicial map $N_\bullet(F_\infty \wr \mathcal{C}_{N,k}) \to N_\bullet(\mathcal{C}_{N,k})$ satisfies the hypothesis of [Seg74, 1.6], so the geometric realization

$$B(F_\infty \wr \mathcal{C}_{N,k}) \to B\mathcal{C}_{N,k}$$

is a quasifibration. Thus for any $(a, p) \in N_0\mathcal{C}_{N,k}$, the inclusion of an actual fiber over (a, p) into the homotopy fiber is a weak equivalence. The actual fiber over $(0, 0)$ is $F_\infty((0, 0)) \simeq D_{N,k-1}$ and the homotopy fiber is equivalent to $\Omega_{(0,0)}B\mathcal{C}_{N,k}$. $\square$

4.1.2. *Second proof.* The sheaf Φ is an example of an *equivariant, continuous sheaf* in the terminology of [Gro86]. This means that Φ is continuously functorial with respect to embeddings (not just inclusions) of open subsets of $\mathbb{R}^N$, cf. proposition 2.11. In particular, $\mathrm{Diff}(U)$ acts continuously on $\Phi(U)$. To such a sheaf on a manifold V there is an associated sheaf Φ^* and a map of sheaves $\Phi \to \Phi^*$. Up to homotopy, $\Phi^*(V)$ is the space of global sections of the fiber bundle $\Phi^{\mathrm{fib}}(TV)$ defined in section 3.4, and the inclusion

$$\Phi(V) \to \Phi^*(V) \simeq \Gamma(V, \Phi^{\mathrm{fib}}(TV)) \tag{4.16}$$

is a scanning map induced by an "exponential" map on V, similar to the map (3.7). Gromov, in [Gro86, section 2.2.2], proves that (4.16) is a weak homotopy equivalence when V is *open*, i.e. all connected components are non-compact, and Φ is *microflexible* (we recall the definition below). This also holds in a relative setting $(V, \partial V)$. In particular we can use $(V, \partial V) = (\mathbb{R} \times I^{N-1}, \mathbb{R} \times \partial I^{N-1})$, in which case (4.16) specializes to (4.12).

That the sheaf Φ is microflexible means that for each inclusion of compact subsets $K' \subseteq K \subseteq \mathbb{R}^N$, each open U, U' with $K' \subseteq U' \subseteq U \supseteq K$, and each diagram

$$\begin{tikzcd} P \times \{0\} \arrow[r, "h"] \arrow[d] & \Phi(U') \arrow[d] \\ P \times [0,1] \arrow[r, "f"] & \Phi(U) \end{tikzcd} \tag{4.17}$$

with P a compact polyhedron, there exists an $\varepsilon > 0$ and an initial lift $P \times [0, \varepsilon] \to \Phi(U')$ of f extending h, after possibly shrinking $U \supseteq K$ and $U' \supseteq K'$.

In this subsection we prove that the sheaf of graphs is microflexible. Then Gromov's h-principle implies that the map (4.12) above is an equivalence for all N.

Proposition 4.14. *Let $K \subseteq U$ be compact and P a polyhedron. Let $f : P \times [0,1] \to \Phi(U)$ be continuous. Then there exists an $\varepsilon > 0$ and a continuous map $g : P \times [0, \varepsilon] \to \Phi(U)$ with the following properties.*

(i) The map $f|P \times [0, \varepsilon]$ agrees with g near K,
(ii) the map $g|P \times \{0\}$ agrees with $f|P \times \{0\}$,
(iii) there exists a compact subset $C \subseteq U$ such that the map

$$P \times [0, \varepsilon] \xrightarrow{g} \Phi(U) \xrightarrow{\mathrm{res}} \Phi(U - C) \tag{4.18}$$

factors through the projection $\mathrm{pr} : P \times [0, \varepsilon] \to P$.

Proposition 4.14 immediately implies microflexibility. Indeed, given maps as in diagram (4.17), the composition

$$P \times [0,\varepsilon] \xrightarrow{\mathrm{pr}} P \times \{0\} \xrightarrow{h} \Phi(U') \to \Phi(U' - C)$$

will agree with $g : P \times [0,\varepsilon] \to \Phi(U)$ on the overlap $U \cap (U' - C) = U - C$, so they can be glued together to a map $P \times [0,\varepsilon] \to \Phi(U')$. The glued map is the initial lift in diagram (4.17).

Proof for P a point. We are given a continuous path $f : [0,1] \to \Phi(U)$. Let $C \subseteq U$ be compact with $K \subseteq \operatorname{int}(C)$ and choose $\tilde{\tau} : U \to [0,1]$ with $\tilde{\tau} = 1$ near K and with $\operatorname{supp}(\tilde{\tau}) \subseteq C$ compact. For each of the finitely many vertices $q \in \mathscr{V}(f(0)) \cap (\operatorname{supp}(\tilde{\tau}) - K)$, choose a function $\rho_q : U \to [0,1]$ which is 1 near q, such that the sets $\operatorname{supp}(\rho_q)$ have compact support in $U - K$ and are mutually disjoint. Let $\tau : U \to [0,1]$ be the function

$$\tau(v) = \tilde{\tau}(v) + \sum_q \rho_q(v)\big(\tilde{\tau}(v) - \tilde{\tau}(q)\big).$$

Then $\tau : U \to [0,1]$ is locally constant outside a compact subset of $U - (K \cup \mathscr{V}(f(0)))$.

Continuity of f gives a graph epimorphism $\varphi_t : f(t) \dashrightarrow f(0)$ for t sufficiently close to 0, defined and canonical near C. Let $g(t)$ be the image of the map

$$\begin{aligned} f(t) &\to U \\ x &\mapsto \tau(x)x + (1 - \tau(x))\varphi(x). \end{aligned}$$

For $t \in [0,1]$ sufficiently close to 0, this defines an element $g(t) \in \Phi(U)$ satisfying (i), (ii), (iii). $\square$

General case. To make the above argument work in the general case (parametrized by a compact polyhedron P), we need only explain how to choose the function $\tau : P \times U \to [0,1]$. For each $p \in P$, the above construction provides a $\tau_p : U \to [0,1]$ that works for $f|\{p\} \times [0,1]$ (i.e. $\tau_p(x,u)$ is independent of u near vertices of $f(p,0)$). The same τ_p will work for $f|\{q\} \times [0,1]$ for all q in a neighborhood $W_p \subseteq P$ of p. Choose a partition of unity $\lambda_p : P \to [0,1]$ subordinate to the open covering by the W_p. Then let

$$\tau(q,v) = \sum_p \lambda_p(q)\tau_p(v). \qquad \square$$

4.2. **The positive boundary subcategory.** The condition on morphisms in the positive boundary subcategory $\mathcal{C}_N^\partial \subseteq \mathcal{C}_N$ (definition 4.5) ensures that any morphism $G : (a_0, A_0, \lambda_0) \to (a_1, A_1, \lambda_1)$ is connected when $|A_1| = 1$. This will allow us to use homological stability to prove the "group completion" result in proposition 4.16 using [MS76], much as it was done in the parallel case of two-dimensional manifolds, [Til97].

Lemma 4.15. *Let $c_0 = (a_0, A_0, \lambda_0)$ and $c_1 = (a_1, A_1, \lambda_1)$ be two objects of $\mathcal{C}_\infty^\partial$, with $a_0 < a_1$ and $|A_1| = 1$. Then*

$$\mathcal{C}_\infty^\partial(c_0, c_1) \simeq \coprod_n BA_n^{1+|A_0|}.$$

Proof. The surjectivity of $A_1 \to \pi_0(G)$ implies that G is connected. Then the lemma follows from theorem 4.3. $\square$

Proposition 4.16. *There is a homology equivalence*

$$\mathbb{Z} \times B\mathrm{Aut}_\infty \to \Omega B\mathcal{C}_\infty^\partial.$$

Proof. This is very similar to lemma 4.13(iii), but with homology equivalences instead of weak homotopy equivalences. We sketch the proof. See [GMTW06, chapter 7] for more details.

Let $A \subseteq \mathrm{int}(I^N)$ be a one-point set. For $n \in \mathbb{N} \subseteq \mathbb{R}$, let

$$F_n : (\mathcal{C}_\infty^\partial)^{\mathrm{op}} \to \mathrm{Spaces}$$

be the functor $F_n = \mathcal{C}_\infty^\partial(-, (n, A, 0))$. $(F_n \wr \mathcal{C}_\infty^\partial)$ has $\mathrm{id}_{(n,A,0)}$ as final object, so $B(F_n \wr \mathcal{C}_\infty^\partial)$ is contractible. Choose morphisms $G_n : (n, A, 0) \to (n+1, A, 0)$ in $\mathcal{C}_\infty^\partial$ with first Betti number $b_1(G_n) = 1$. This defines a direct system

$$F_1 \xrightarrow{G_1} F_2 \xrightarrow{G_2} \dots \xrightarrow{G_n} F_n \xrightarrow{G_{n+1}} \dots$$

and we let $F_\infty(x) = \mathrm{hocolim}_n F_n(x)$. Then $B(F_\infty \wr \mathcal{C}_\infty^\partial) = \mathrm{hocolim}_n B(F_n \wr \mathcal{C}_\infty^\partial)$ is still contractible.

Lemma 4.15 gives a homotopy equivalence for each object $c = (a, A_0, \lambda)$

$$F_\infty(c) \simeq \mathbb{Z} \times BA_\infty^{1+|A_0|}$$

and by theorem 1.4, the functor $F_\infty : (\mathcal{C}_\infty^\partial)^{\mathrm{op}} \to \mathrm{Spaces}$ maps every morphism to a homology equivalence. This implies that the simplicial map $N_\bullet(F_\infty \wr \mathcal{C}_\infty^\partial) \to N_\bullet \mathcal{C}_\infty^\partial$ satisfies the assumption of [MS76, proposition 4] and therefore that the geometric realization

$$B(F_\infty \wr \mathcal{C}_\infty^\partial) \to B\mathcal{C}_\infty^\partial \tag{4.19}$$

is a homology fibration in the sense of [MS76]. Thus for any $(a, A, \lambda) \in N_0\mathcal{C}_\infty^\partial$, the inclusion of an actual fiber over (a, A) of (4.19) into the homotopy fiber is a homology equivalence. The actual fiber over $(0, \emptyset, 0)$

is $F_\infty((0,\emptyset,0)) \simeq \mathbb{Z} \times B\mathrm{Aut}_\infty$. Since $B(F_\infty \wr \mathcal{C}_\infty^\partial)$ is contractible, the homotopy fiber is equivalent to $\Omega_{(0,\emptyset,0)} B\mathcal{C}_\infty^\partial$. □

The following proposition is proved in several steps. The proof occupies the rest of this section, and is similar to [GMTW06, chapter 6].

Proposition 4.17. *The inclusion induces a weak equivalence*

$$B\mathcal{C}_\infty^\partial \xrightarrow{\simeq} B\mathcal{C}_\infty.$$

For $G \in D_N$, we shall write $f_G : G \to \mathbb{R}$, or just f, for the restriction to G of the projection $\mathbb{R} \times \mathrm{int}(I^{N-1}) \to \mathbb{R}$.

Definition 4.18. Let $G \in D_N$ and $p \in f^{-1}((-\infty, 0])$. Define $f^-(p) \in [-\infty, f(p)]$ as

$$f^-(p) = \max_\gamma \min_{t \in [0,1]} f\gamma(t)$$

where the maximum is taken over paths $\gamma : [0,1] \to G$ satisfying $\gamma(0) = p$ and $f\gamma(1) > 0$. We let $f^-(p) = -\infty$ if no such γ exists. Let $A_G \subseteq f^{-1}((-\infty, 0])$ be the closure of the set of points for which $f^-(p) < f(p)$. Let $B_G \subseteq G$ be the union of A_G, the set of vertices, and the set of edgewise critical points of f. Let $R_G = (-\infty, 0] - f(B_G)$ and

$$D_N^\partial = \{G \in D'_N \mid R_G \neq \emptyset\}.$$

For fixed $r \in (-\infty, 0]$, the set $\{G \in D_N \mid r \in R_G\}$ is an open subset of D_N.

Lemma 4.19. *There is a weak equivalence $D_N^\partial \simeq B\mathcal{C}_N^\partial$.*

Proof. This is completely analogous to the proof of theorem 4.7 in section 4.1. It uses the subposet $D_N^{\partial,\pitchfork}$ of $D_N^\pitchfork$ consisting of (a, G) with $G \in D_N^\partial$ and $a \in R_G$ and the poset $D_N^{\partial,\perp} = D_N^\perp \cap D_N^{\partial,\pitchfork}$. As in the proof of theorem 4.7 we have levelwise equivalences

$$N_\bullet D_N^{\partial,\perp} \xrightarrow{\simeq} N_\bullet D_N^{\partial,\pitchfork}, \quad N_\bullet D_N^{\partial,\perp} \xrightarrow{\simeq} N_\bullet \mathcal{C}_N^\partial,$$

and the equivalence $BD_N^{\partial,\pitchfork} \to D_N^\partial$ uses lemma 3.4. □

Proving proposition 4.17 now amounts to the inclusion $D_N^\partial \subset D_N$ being a weak equivalence. This is done in the lemmas 4.21 and 4.24 below.

Definition 4.20. Let $D'_N \subseteq D_N$ be the subset consisting of graphs G for which no path component of G is compact.

Lemma 4.21. *The inclusion $D'_N \to D_N$ is a weak equivalence.*

Proof. For a given $G \in D'_N$, we can assume, after possibly perturbing the function f a little, that no connected component of f is contained in $f^{-1}(0)$. Then we can choose an $\varepsilon > 0$ small enough that no connected component of $f^{-1}((-\varepsilon, \varepsilon)) \subseteq G$ is compact. For $t \in [0,1]$ let $h_t : \mathbb{R} \to \mathbb{R}$ be an isotopy of embeddings with $h_0 = \mathrm{id}$ and $h_1(\mathbb{R}) = (-\varepsilon, \varepsilon)$. Let $H_t = h_t \times \mathrm{id} : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R} \times \mathbb{R}^{N-1}$. Then

$$t \mapsto G_t = H_t^*(G)$$

defines a continuous path $[0,1] \to D_N$, starting at $G_0 = G$ and ending in $G_1 \in D'_N$.

This proves that the relative homotopy group $\pi_k(D_N, D'_N)$ is trivial for $k = 0$. The case $k > 0$ is similar: Given a continuous map of pairs

$$q : (\Delta^k, \partial\Delta^k) \to (D_N, D'_N)$$

we can first perturb q a little, such that for all $x \in \Delta^k$, no connected component of $q(x)$ is contained in $f^{-1}(0)$, and then stretch a small interval $(-\varepsilon, \varepsilon)$. □

The relevance of the condition that no connected component of $G \in D'_N$ be compact lies in the following definition.

Definition 4.22. For $G \in D_N$ let $\hat{G} = G \amalg \{+\infty, -\infty\}$. Then f extends to $f : \hat{G} \to [-\infty, \infty]$, and we equip $\hat{G}$ with the coarsest topology in which $G \subseteq \hat{G}$ has the subspace topology and $f : \hat{G} \to [-\infty, \infty]$ is continuous. (In other words, a sequence of points $x_n \in G$, $n \in \mathbb{N}$, converges to $\pm\infty \in \hat{G}$ if and only if $f(x_n) \to \pm\infty$.) An *escape to* $+\infty$ is a path $\gamma : [0,1] \to \hat{G}$ such that $\gamma(0) = p$ and $\gamma(1) = +\infty$. An escape to $-\infty$ is defined similarly.

Given G and p, an escape to either $+\infty$ or $-\infty$ exists if and only if the path component of G containing p is non-compact. Let us also point out that a path $\gamma : [0,1] \to \hat{G}$ is uniquely given by its restriction $[0,1] \dashrightarrow G$, defined on $\gamma^{-1}(G) \subseteq [0,1]$.

Remark 4.23. The statement of lemma 4.21 is that any map of pairs $q : (\Delta^k, \partial\Delta^k) \to (D_N, D'_N)$ is homotopic to a map q' such that for any $x \in \Delta^k$ there exists an escape to $\pm\infty$ from $p \in q'(x)$. In fact, essentially the same proof gives a slightly stronger statement, namely that such escapes exist locally in Δ^k (not just pointwise).

Indeed, if $p \in f^{-1}((-\varepsilon, \varepsilon))$ and $\gamma : [0,1] \to G$ is a path with $\gamma(0) = p$ and $|f\gamma(1)| > \varepsilon$ and H_t is the isotopy from the proof of lemma 4.21, then $H_1^{-1} \circ \gamma : [0,1] \dashrightarrow G_1 = H_1^*(G)$ is an escape from $H_1^{-1}(p)$ to either $+\infty$ or to $-\infty$. If $G = q(x_0)$ for some $x_0 \in \Delta^k$, then the path γ can be extended locally to $\Gamma : U_x \times [0,1] \to \mathbb{R}^N$ for a neighborhood $U_x \subseteq \Delta^k$

of x, such that $\Gamma(x,t) \in q(x)$ and $\Gamma(x_0,-) = \gamma$. Then $H_1^{-1} \circ \Gamma$ is a family of escapes to $+\infty$ or $-\infty$, defined locally near x_0.

Lemma 4.24. *The inclusion $D_\infty^\partial \to D'_\infty$ is a weak homotopy equivalence.*

Proof. We prove that for $k \geq 0$, any map of pairs

$$q : (\Delta^k, \partial\Delta^k) \to (D'_\infty, D_\infty^\partial) \tag{4.20}$$

is homotopic to a map into D_∞^∂.

Consider first the case $k = 0$. Let $G = q(1)$. Choose $a, b \in \mathbb{R}$ with $a < 0 < b$ and $G \pitchfork \{a, b\} \times \mathbb{R}^\infty$. If G satisfies the condition that

$$\pi_0(f^{-1}(b)) \to \pi_0(f^{-1}([a,b])) \text{ is surjective} \tag{4.21}$$

then $[a, a+\varepsilon] \subseteq R_G$ for some $\varepsilon > 0$, and hence $G \in D_\infty^\partial$. For general $G \in D'_\infty$ we will construct a path $h : [0,1] \to D'_\infty$ with $h(0) = G$ and such that $h(1)$ satisfies (4.21).

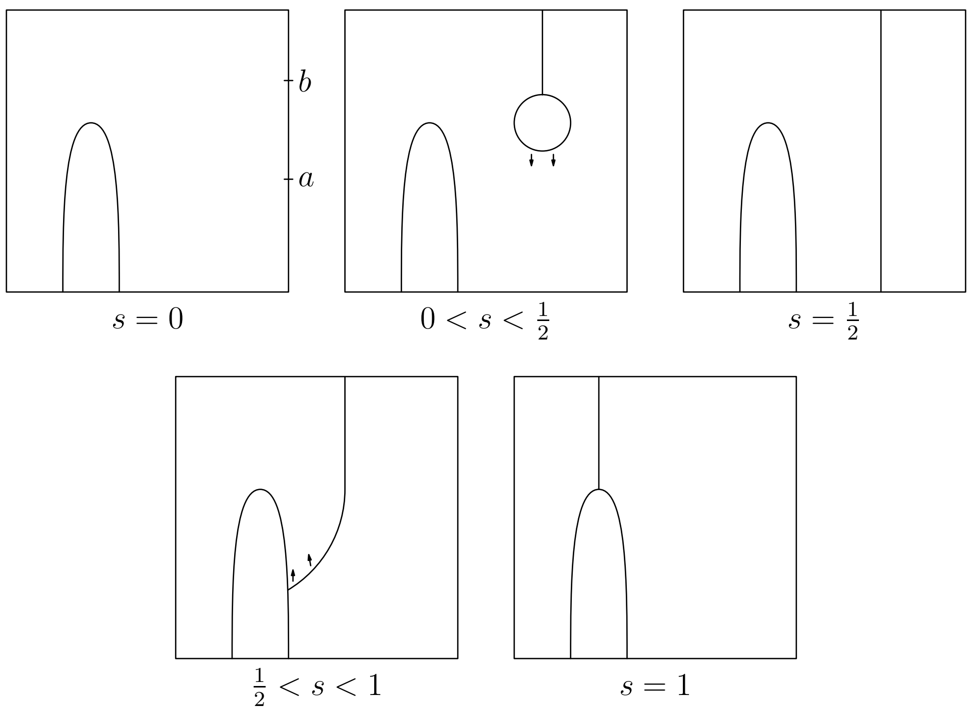

FIGURE 3. $h(s)$ for various $s \in [0,1]$

Let $p \in f^{-1}([a,b])$, and let $\gamma : [0,1] \dashrightarrow G$ be an escape from p to $-\infty$. A typical such G is depicted in the first picture in the cartoon in figure 3 which also depicts a path $h = h_\gamma : [0,1] \to D'_\infty$. The

pictures show part of the graph $h(s) \in D'_\infty$ for various $s \in [0,1]$. $f_{h(s)} : h(s) \to \mathbb{R}$ is the height function (projection onto vertical axis) in the pictures. At time $s = 0$ the graph $G = h(0)$ has a local maximum in $f^{-1}([a,b])$. At times $s \in (0, \frac{1}{2})$, the graph $h(s)$ is the disjoint union of G and a "gallow". At times $s \in [\frac{1}{2}, 1]$, $h(s)$ is obtained from G by attaching an extra edge at the point $\gamma(2-2s) \in \hat{G}$. The path h depends on two choices. Most importantly, it depends on the escape γ from p to $-\infty$ along which to "slide" the attached extra edge. Secondly, we have only defined the graph $h(s)$ abstractly; to get an element of D'_∞, we choose an embedding $h(s) \subset \mathbb{R} \times \mathrm{int}(I^{N-1})$ extending the inclusion of $G \subset h(s)$. Such an embedding always exists when $N = \infty$, so we suppress it from the notation.

The path h_γ has two convenient properties. Firstly we have the inclusion $G \subseteq h(s)$ for all $s \in [0,1]$. As constructed in the cartoon in figure 3, all local maxima of $f_{h(s)} : h(s) \to \mathbb{R}$ are in $G \subseteq h(s)$, so the subset

$$R_G \cap R_{h(s)} \subseteq R_G$$

is open and dense. In particular $R_{h(s)}$ is non-empty if R_G is non-empty. So the path $h : [0,1] \to D'_\infty$ runs entirely in D^∂_∞, provided $h(0) \in D^\partial_\infty$. Secondly, at time $s = 1$, the graph $G_1 = h(1)$ is obtained from G by attaching an extra edge extending from $p \in G$ to $+\infty$. This assures that if $x \in f^{-1}([a,b])$ is in the same path component as p, then there is an escape from x to $+\infty$ which stays inside $f^{-1}_{G_1}([a,\infty))$. In particular $f^-_{G_1}(x) \geq a$. If x is sufficiently close to $f^{-1}(a)$, then by transversality we will even have $f^-_{G_1}(x) = x$.

There is a similar construction if $\gamma : [0,1] \to \hat{G}$ is an escape from p to $+\infty$, only easier: Let $h_\gamma(s)$ be the graph obtained from G by attaching an extra edge extending to $+\infty$ at the point $\gamma(1-s) \in \hat{G}$.

The same construction can be applied to attach several extra edges at the same time. Let $X \subseteq f^{-1}_G((a,b))$ be a finite subset and let $\Gamma : X \times [0,1] \to \hat{G}$ be such that $\Gamma(p, -)$ is an escape from p to $\pm\infty$. Then the above construction gives a path $h = h_\Gamma : [0,1] \to D'_\infty$ such that $h_\Gamma(1)$ is obtained from $G = h_\Gamma(0)$ by attaching an extra edge extending to $+\infty$ at all the points $p \in X$. If $X \subseteq f^{-1}((a,b))$ is chosen such that the inclusion

$$X \amalg f^{-1}(b) \to f^{-1}([a,b])$$

induces a surjection in π_0, then the resulting graph $G_1 = h(1)$ will satisfy $f^-_{G_1}(x) = x$ for all x in a sufficiently small neighborhood of

$f^{-1}(a)$, and hence $R_{G_1} \neq \emptyset$, so $h(1) \in D^\partial_\infty$. This finishes the proof for $k = 0$.

For $k > 0$ we will give a parametrized version of the above argument. If $\Gamma : X \times [0,1] \to \hat{G}$ and $\Gamma' : X' \times [0,1] \to \hat{G}$ are two sets of escapes to $\pm\infty$, then we get two paths $h = h_\Gamma : [0,1] \to D'_\infty$ and $h' = h_{\Gamma'} : [0,1] \to D'_\infty$. We will say that h and h' are *compatible* if the inclusion

$$G \subseteq h(s) \cap h'(s')$$

is an equality for all $s, s' \in [0,1]$. In this case we will have that $h(s) \cup h'(s')$ is the pushout of the diagram $h'(s') \leftarrow G \to h(s)$ and that $h(s) \cup h'(s') \in D'_\infty$. This produces a continuous map

$$\begin{aligned} [0,1]^2 &\to D'_\infty \\ (s,s') &\mapsto h(s) \cup h'(s'). \end{aligned}$$

More generally a finite set of escapes $\Gamma_j : X_j \times [0,1] \to \hat{G}$, $j \in J$ produces a finite set of paths $h_j : [0,1] \to D'_\infty$, and their union gives a map

$$\begin{aligned} h_J : [0,1]^J &\to D'_\infty \\ (s_j)_{j\in J} &\mapsto \cup_j h(s_j), \end{aligned}$$

provided the h_j are *pairwise compatible*, i.e. that h_i and h_j are compatible for all $i, j \in J$ with $i \neq j$.

We will have $h_J(t) \in D^\partial_\infty$ as long as *at least* one of the coordinates of $t \in [0,1]^J$ is 1. This is an important property of the homotopies constructed from the cartoon: If we have already attached enough extra edges to ensure $G \in D^\partial_\infty$, then attaching even more edges will not destroy the property of being in D^∂_∞ (even if we stop in the middle of the attaching process).

Now let $k \geq 1$, and let q be as in (4.20). We will use the above construction to prove that q is homotopic to a map into D^∂_∞, and hence that $\pi_k(D'_\infty, D^\partial_\infty)$ vanishes. If $x \in \Delta^k$ has $q(x) \notin D^\partial_\infty$, then the proof in the case $k = 0$ gives a path $h = h_\Gamma$ from $q(x)$ to a point in D^∂_∞, depending on a family Γ of escapes to $\pm\infty$. Extending Γ to a continuous family $\Gamma_y : X \times [0,1] \to \widehat{q(y)}$, $y \in U_x$, parametrized by a neighborhood U_x of x, we get a homotopy

$$h_x : U_x \times [0,1] \to D'_\infty$$

starting at $q|U_x$ and ending in a map $U_x \to D^\partial_\infty$. The extension of Γ to a continuous family Γ_y, $y \in U_x$, can be assumed to exist by remark 4.23. Thus we get an open covering of Δ^k by the sets U_x, $x \in \Delta^k$, and

corresponding homotopies h_x. We now explain how to glue all these together.

Choose a triangulation K of Δ^k so fine that for all $v \in \mathrm{Vert}(K)$ we have $\mathrm{st}(v) \subseteq U_x$ for some $x \in \Delta^k$. Let $\mathrm{sd}K$ denote the barycentric subdivision of K. For $\sigma \in \mathrm{Vert}(\mathrm{sd}K) = \mathrm{Simp}(K)$ we write $\mathrm{st}(\sigma)$ for the open star of σ as a 0-simplex of $\mathrm{sd}(K)$. We write $\dim(\sigma)$ for the dimension as a simplex of K, and $\tau < \sigma$ if τ is a proper face (in K) of σ. Choose bump functions $\lambda_\sigma : |K| \to [0,1]$, $\sigma \in \mathrm{Vert}(\mathrm{sd}K)$ with the properties

(i) $\mathrm{supp}(\lambda_\sigma) \subseteq \mathrm{st}(\sigma)$ for all σ,
(ii) $|K| = \cup_\sigma \mathrm{int}(\lambda_\sigma^{-1}(1))$.

Then we have $\mathrm{supp}(\lambda_\sigma) \cap \mathrm{supp}(\lambda_\tau) = \emptyset$ unless $\tau < \sigma$ (or $\sigma = \tau$ or $\sigma < \tau$). For a simplex $\chi = (\sigma_0 < \sigma_1 < \dots < \sigma_l) \in \mathrm{Simp}(\mathrm{sd}K)$, we have a corresponding geometric simplex

$$|\chi| \subseteq |\mathrm{sd}K| = |K|$$

and λ_τ vanishes on this subspace unless τ is a vertex of χ.

For each $\sigma \in \mathrm{Vert}(\mathrm{sd}K)$, choose an $x \in \Delta^k$ with $\mathrm{st}(\sigma) \subseteq U_x$. Let $h_\sigma = h_x|(\mathrm{st}(\sigma) \times [0,1])$. This gives a homotopy

$$h_\sigma : \mathrm{st}(\sigma) \times [0,1] \to D'_\infty,$$

starting at $g|\mathrm{st}(\sigma)$ and ending in a map $\mathrm{st}(\sigma) \to D^\partial_\infty$. Proceeding by induction on $\dim(\sigma)$, we can assume that h_σ is compatible with h_τ over $\mathrm{supp}(\lambda_\sigma) \cap \mathrm{supp}(\lambda_\tau)$ for all faces $\tau < \sigma$ (we can assume this because $N = \infty$). Then define H_χ as the composition

$$|\chi| \times [0,1] \to |\chi| \times [0,1]^{l+1} \to D'_\infty,$$

where the first map is given by

$$(x, s) \mapsto (x, (\lambda_{\sigma_0}(x), \dots, \lambda_{\sigma_l}(x))s)$$

and the second is given as $\cup_{i=0}^l h_{\sigma_i}$. This is well-defined because the σ_i's are proper faces of each other, so the homotopies h_{σ_i} are compatible. The homotopies H_χ glue together to a homotopy

$$H : |K| \times [0,1] \to D'_\infty,$$

starting at q and ending in a map $H(-,1) : \Delta^k \to D^\partial_\infty$. □

5. Homotopy type of the graph spectrum

The main result in this section is the following, which will finish the proof of theorem 1.5.

Theorem 5.1. *We have an equivalence of spectra* $\mathbf{\Phi} \simeq S^0$ *and hence a weak equivalence*

$$\Omega^\infty \mathbf{\Phi} \simeq QS^0.$$

Let us first give an informal version of the proof. Since any ε-neighborhood of $0 \in \mathbb{R}^N$ can be stretched to all of $\mathbb{R}^N$, the restriction map

$$\Phi(\mathbb{R}^N) \to \Phi(0 \in \mathbb{R}^N)$$

to the "space" of germs near 0 is an equivalence. Now, a germ of a graph around a point is easy to understand: Either it is the empty germ, or it is the germ of a line through the point, or it is the germ of $k \geq 3$ half-lines meeting at the point. Any non-empty germ is essentially determined by $k \geq 2$ points on S^{N-1}, so the space of non-empty germs of graphs is essentially the space of finite subsets of cardinality ≥ 2 of S^{N-1}. Let $\mathrm{Sub}(S^{N-1})$ denote the space of non-empty finite subsets of S^{N-1}. The space $\mathrm{Sub}(S^{N-1})$ is not quite right, for two reasons—it doesn't model the empty germ, and it includes points that it shouldn't, namely the space of 1-point subsets $S^{N-1} \subseteq \mathrm{Sub}(S^{N-1})$. Both of these problems can be fixed by collapsing the space of 1-point subsets $S^{N-1} \subseteq \mathrm{Sub}(S^{N-1})$ to a point. The above discussion defines a map

$$\Phi(0 \in \mathbb{R}^N) \to \mathrm{Sub}(S^{N-1})/S^{N-1}, \tag{5.1}$$

which maps the empty germ to $[S^{N-1}]$ and maps the germ of $(G, 0)$ to the set of tangent directions of G at 0. It seems reasonable that this map should be a homotopy equivalence (it even seems close to being a homeomorphism: If we had considered instead piecewise linear graphs, it would be a bijection). Curtis and To Nhu [CTN85] proves that $\mathrm{Sub}(S^{N-1})$ is contractible. (In fact they prove that it is homeomorphic to $\mathbb{R}^\infty$. For an easy, and more relevant, proof of weak contractibility see [Han00] or [BD04, §3.4.1].) Therefore the right hand side of the map (5.1) is homotopy equivalent to S^N as we want. Unfortunately, the natural map from $\Phi(\mathbb{R}^N)$ to the right hand side of (5.1), which assigns to $G \in \Phi(\mathbb{R}^N)$ the set of directions of half-edges through $0 \in \mathbb{R}^N$, is not even continuous.

Let $\mathscr{D}$ be the category of finite sets and surjections. Then

$$\mathrm{Sub}(S^{N-1}) = \operatorname*{colim}_{T \in \mathscr{D}^{\mathrm{op}}} \prod_T S^{N-1}.$$

A step towards rectifying (5.1) to a continuous map is to replace the colimit by the homotopy colimit. But the real reason for discontinuity is that from the point of view of germs at a point, the collapse of an edge leads to a *sudden* splitting of one half-edge into two. To fix

this, we will fatten up $\Phi(\mathbb{R}^N)$ in a way that allows us to remove the suddenness of edge collapses, and remotely similar to the proof of the equivalence (4.9) in section 4.1.

5.1. **A pushout diagram.** The main result of this section is proposition 5.4 below. Recall that a graph G is a tree if it is contractible (in particular non-empty).

Definition 5.2. Let $\mathscr{C}$ be the topological category whose objects are triples (G, r, φ), where $G \in \Phi(\mathbb{R}^N)$ and $r > 0$ satisfies that $G \pitchfork \partial B(0,r)$ and that $G \cap B(0,r)$ is a tree. φ is a labelling of the set of leaves, i.e. a bijection

$$\varphi : \underline{k} = \{1, \dots, k\} \xrightarrow{\cong} G \cap \partial B(0,r).$$

Topologize $\mathrm{ob}(\mathscr{C})$ as a subset

$$\mathrm{ob}(\mathscr{C}) \subseteq U_1 \times \coprod_{\substack{r>0 \\ k \geq 2}} \mathrm{Map}(\underline{k}, \partial B(0,r)).$$

There is a unique morphism $(G, r, \varphi) \to (G', r', \varphi')$ if and only if $G = G'$ and $r \leq r'$, otherwise there is none.

Definition 5.3. Let $E_\bullet$ be the simplicial space where an element of $E_k \subseteq N_k\mathscr{C} \times S^N$ is a pair (χ, p), where $\chi = (G, r_0 < r_1 < \dots < r_k, \{\varphi_i\}) \in N_k\mathscr{C}$ and $p \in S^N = \mathbb{R}^N \cup \{\infty\}$ satisfies

$$p \in \mathbb{R}^N \cup \{\infty\} - \big(G \cap B(0, r_k) - \mathrm{int} B(0, r_0)\big).$$

Include $N_\bullet\mathscr{C} \subset E_\bullet$ as the subset with $p = \infty$.

Proposition 5.4. *Let $B\mathscr{C} \to |E_\bullet|$ be included as the subspace with $p = \infty$. Then we have a weak equivalence $\Phi(\mathbb{R}^N) \simeq |E_\bullet|/B\mathscr{C}$.*

Proposition 5.4 is proved in several steps. First, in lemma 5.6, we write $\Phi(\mathbb{R}^N)$ as a homotopy pushout of three open subsets U_0, U_1, and U_{01}. In lemma 5.8 we give a similar description of $|E_\bullet|/B\mathscr{C}$ as a homotopy pushout. Then we relate the homotopy pushout diagrams by a zig-zag of weak equivalences maps according to diagram (5.6).

Definition 5.5.

(i) Let $U_0 \subseteq \Phi(\mathbb{R}^N)$ be the subset consisting of graphs G satisfying $0 \notin G$.

(ii) Let $U_1 \subseteq \Phi(\mathbb{R}^N)$ be the subset consisting of graphs G for which there exists an $r > 0$ such that $G \pitchfork \partial B(0,r)$ and that $G \cap B(0,r)$ is a tree.

(iii) Let $U_{01} = U_0 \cap U_1$.

Lemma 5.6. *The homotopy pushout (double mapping cylinder) of the diagram*

$$U_0 \leftarrow U_{01} \rightarrow U_1 \tag{5.2}$$

is weakly equivalent to $\Phi(\mathbb{R}^N)$.

Proof. $\Phi(\mathbb{R}^N)$ is the union of the two subsets U_0 and U_1, and it is easy to see that both of these are open. □

To give a similar pushout description of $|E_\bullet|/B\mathscr{C}$ in lemma 5.8 below we need the following definitions.

Definition 5.7. Let F_0, F_{01}, and F_1 be the functors $\mathscr{C} \rightarrow$ Spaces given by

$$\begin{aligned} F_0(G, r, \varphi) &= \mathbb{R}^N \cup \{\infty\} - G \cap B(0, r) \\ F_{01}(G, r, \varphi) &= \mathrm{int} B(0, r) - G \\ F_1(G, r, \varphi) &= \mathrm{int} B(0, r). \end{aligned}$$

F_0 is contravariant and F_{01} and F_1 are covariant. All three spaces $N_k(F_0 \wr \mathscr{C})$, $N_k(\mathscr{C} \wr F_{01})$ and $N_k(\mathscr{C} \wr F_1)$ are open subsets of $N_k\mathscr{C} \times S^N$, where $S^N = \mathbb{R}^N \cup \{\infty\}$.

Lemma 5.8. $|E_\bullet|$ *is weakly equivalent to the homotopy pushout of the diagram*

$$B(F_0 \wr \mathscr{C}) \leftarrow B(\mathscr{C} \wr F_{01}) \rightarrow B(\mathscr{C} \wr F_1), \tag{5.3}$$

and $|E_\bullet|/B\mathscr{C}$ *is weakly equivalent to the homotopy pushout of the diagram*

$$B(F_0 \wr \mathscr{C})/B\mathscr{C} \leftarrow B(\mathscr{C} \wr F_{01}) \rightarrow B(\mathscr{C} \wr F_1). \tag{5.4}$$

Proof. As subsets of $N_k\mathscr{C} \times S^N$ we have

$$\begin{aligned} N_k(F_0 \wr \mathscr{C}) \cap N_k(\mathscr{C} \wr F_1) &= N_k(\mathscr{C} \wr F_{01}) \\ N_k(F_0 \wr \mathscr{C}) \cup N_k(\mathscr{C} \wr F_1) &= E_k. \end{aligned}$$

Then E_k is weakly equivalent to the homotopy pushout of the following diagram

$$N_k(F_0 \wr \mathscr{C}) \leftarrow N_k(\mathscr{C} \wr F_{01}) \rightarrow N_k(\mathscr{C} \wr F_1). \tag{5.5}$$

But the homotopy pushout of diagram (5.3) is homeomorphic to the geometric realization of the simplicial space whose k-simplices is the homotopy pushout of (5.5). The second part is similar. □

We will now relate the pushout diagram (5.2) to the pushout diagram (5.4) by a zig-zag of maps, according to the following diagram.

$$\begin{tikzcd}
U_0 & U_{01} \arrow[l] \arrow[r] & U_1 \\
U_0 \arrow[u, equal] \arrow[d] & B\mathscr{C}_{01} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B\mathscr{C} \arrow[u] \arrow[d] \\
* & B(\mathscr{C} \wr F_{01}) \arrow[l] \arrow[r] & B(\mathscr{C} \wr F_1) \\
B(F_0 \wr \mathscr{C})/B\mathscr{C} \arrow[u] & B(\mathscr{C} \wr F_{01}) \arrow[l] \arrow[r] \arrow[u, equal] & B(\mathscr{C} \wr F_1). \arrow[u, equal]
\end{tikzcd} \tag{5.6}$$

The spaces and maps in the diagram will be defined below, and we will prove that all vertical maps are weak equivalences. We first consider the second row of the diagram.

Definition 5.9. Let $\mathscr{C}_{01}$ be the subcategory of $\mathscr{C}$ consisting of (G, r, φ) with $G \in U_{01}$.

Proposition 5.10. *The forgetful maps*

$$B\mathscr{C} \to U_1, \quad B\mathscr{C}_{01} \to U_{01}$$

are both weak equivalences.

Proof. This is completely similar to theorem 4.7 and lemma 4.12: $N_k\mathscr{C} \to U_1$ is etale for all k, and for $G \in U_1$, the inverse image in $\mathscr{C}$ is equivalent as a category to a totally ordered non-empty set. Similarly for $B\mathscr{C}_{01} \to U_{01}$. $\square$

Maps from the second to the third row in diagram (5.6) are induced by the natural diagram of functors

$$\begin{tikzcd}
\mathscr{C}_{01} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
\mathscr{C} \wr F_{01} \arrow[r] & \mathscr{C} \wr F_1.
\end{tikzcd} \tag{5.7}$$

The horizontal functors in (5.7) are the natural inclusions, and the vertical functors are both given by $(G, r, \varphi) \mapsto (G, r, \varphi, 0)$.

Lemma 5.11.

(i) U_0 *is contractible.*
(ii) $B\mathscr{C} \to B(\mathscr{C} \wr F_1)$ *is a weak equivalence.*
(iii) $B\mathscr{C}_{01} \to B(\mathscr{C} \wr F_{01})$ *is a weak equivalence.*

Proof. (i) follows by pushing radially away from $p = 0$, as in lemma 2.6.

(ii) is also easy. Moving the point $p \in \mathrm{int}B(0, r)$ to 0 along a straight line defines a deformation retraction of $B(\mathscr{C} \wr F_1)$ onto the image of $B\mathscr{C}$.

For (iii), notice that for each k we have the following pullback diagram of spaces

$$\begin{tikzcd} N_k\mathscr{C}_{01} \arrow[r] \arrow[d] & N_k(\mathscr{C} \wr F_{01}) \arrow[d] \\ \coprod_{r>0}\{0\} \arrow[r] & \coprod_{r>0} \mathrm{int}B(0, r). \end{tikzcd}$$

It is easy to see that the right hand vertical map is a fibration (in fact a trivial fiber bundle), so the diagram is also homotopy pullback. The bottom horizontal map is obviously a homotopy equivalence, so it follows that $N_k\mathscr{C}_{01} \to N_k(\mathscr{C} \wr F_{01})$ is an equivalence for all k. This proves (iii). $\square$

The map from the third to the fourth row of diagram (5.6) is covered by the following lemma.

Lemma 5.12. *The inclusion $\{\infty\} \to F_0(G, r, \varphi)$ is a homotopy equivalence, and $B(F_0 \wr \mathscr{C})/B\mathscr{C}$ is weakly contractible.*

Proof. $N_k(F_0 \wr \mathscr{C})$ is an open subset of $N_k\mathscr{C} \times S^N$ such that all fibers of the projection

$$N_k(F_0 \wr \mathscr{C}) \to N_k\mathscr{C}$$

are contractible. It follows from [Seg78, proposition (A.1)] that the projection is a Serre fibration and hence a weak equivalence. Therefore the section $N_k\mathscr{C} \to N_k(F_0 \wr \mathscr{C})$ obtained by setting $p = \infty$ is also a weak equivalence. It is easy to see that this section is a cofibration, so the quotient

$$N_k(F_0 \wr \mathscr{C})/N_k\mathscr{C}$$

is weakly contractible. $\square$

This finishes the proof of proposition 5.4.

Remark 5.13. From the third line in diagram (5.6) it follows that $\Phi(\mathbb{R}^N)$ is weakly equivalent to the mapping cone of the map $B(\mathscr{C} \wr F_{01}) \to B\mathscr{C}$. One can think of this map as a "generalized spherical fibration", and hence of the mapping cone as a "generalized Thom space", in the following sense. The fiber of the map

$$N_k(\mathscr{C} \wr F_{01}) \to N_k\mathscr{C}$$

over a point $(G, r_0 < r_1 < \cdots < r_k, \{\varphi_i\})$ is the space

$$\operatorname{int} B(0, r_0) - G \simeq \bigvee^{k_0 - 1} S^{N-2},$$

where k_0 is the cardinality of the set $G \cap \partial B(0, r_0)$. Thus, the fibers of $B(\mathscr{C} \wr F_{01}) \to B\mathscr{C}$ are not spheres, as they would be were the map an honest spherical fibration, but wedges of spheres, where the number of spheres in the fiber varies over the base.

5.2. **A homotopy colimit decomposition.** Let $\mathscr{D}_{\geq 2}$ be the category whose objects are finite sets of cardinality at least 2, and whose morphisms are the surjective maps of sets. In this section we will first rewrite $|E_\bullet|/B\mathscr{C}$ stably as the pointed homotopy colimit of a functor $H : \mathscr{D}_{\geq 2}^{\mathrm{op}} \to \mathrm{Spaces}$. This is done in proposition 5.16 below. Then we prove that this pointed homotopy colimit is weakly equivalent to S^N in proposition 5.23. Together these results prove theorem 5.1.

There is a functor $T : \mathscr{C} \to \mathscr{D}_{\geq 2}^{\mathrm{op}}$ defined in the following way. Let $(G, r, \varphi) \to (G, r', \varphi')$ be a morphism in $\mathscr{C}$. We have a diagram of inclusions

$$G \cap \partial B(0, r) \xrightarrow{i_r} G \cap (B(0, r') - \operatorname{int} B(0, r)) \xleftarrow{i_{r'}} G \cap \partial B(0, r')$$

in which the inclusion i_r is a homotopy equivalence and $i_{r'}$ induces a surjection in π_0.

Definition 5.14. Let $f : (G, r, \varphi) \to (G, r', \varphi')$ be a morphism, and let i_r and $i_{r'}$ be as above. Then let $T(f)$ be the composition

$$\varphi^{-1} \circ (\pi_0 i_r)^{-1} \circ (\pi_0 i_{r'}) \circ \varphi' : \underline{k'} \to \underline{k}.$$

This defines a functor $T : \mathscr{C} \to \mathscr{D}_{\geq 2}^{\mathrm{op}}$.

Definition 5.15. For $\underline{k} \in \mathscr{D}_{\geq 2}^{\mathrm{op}}$, let $\Delta \subseteq (S^{N-1})^{\underline{k}} = \mathrm{Map}(\underline{k}, S^{N-1})$ be the diagonal. Let the functor $H : \mathscr{D}_{\geq 2}^{\mathrm{op}} \to \mathrm{Spaces}$ be the quotient

$$H(\underline{k}) = \mathrm{Map}(\underline{k}, S^{N-1})/\Delta.$$

The following proposition will be proved below in several steps. We will say that a map is "highly connected" if it is $c(N)$-connected for a function $c : \mathbb{N} \to \mathbb{N}$ such that $c(N) \to \infty$ as $N \to \infty$. Similarly we will say that a map is "N+highly connected" if it is $(N + c(N))$-connected.

Proposition 5.16. *There is an N+highly connected map*

$$|E_\bullet|/B\mathscr{C} \to B(\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr H)/B\mathscr{D}_{\geq 2}^{\mathrm{op}}.$$

Recall that the space $B(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr H)$ is the homotopy colimit of H. Each $H(\underline{k})$ has the basepoint $[\Delta]$ which defines an inclusion $B\mathscr{D}^{\mathrm{op}}_{\geq 2} \subset B(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr H)$. The quotient space is the *pointed* homotopy colimit of the functor H.

Let $K \subseteq \mathbb{R}^N$ be a compact subset with contractible path components. The *duality* map is the map

$$A : (\mathbb{R}^N - K) \to \mathrm{Map}(K, S^{N-1})$$

given by

$$A(p)(x) = \frac{p - x}{|p - x|}$$

The map A is $(2N-3)$-connected. Indeed, it is homotopy equivalent to the inclusion

$$\bigvee^{\pi_0 K} S^{N-1} \to \prod_{\pi_0 K} S^{N-1}.$$

Let $\Delta \subseteq \mathrm{Map}(K, S^{N-1})$ denote the constant maps. A induces a well defined, continuous map

$$\mathbb{R}^N \cup \{\infty\} - K \xrightarrow{A} \mathrm{Map}(K, S^{N-1})/\Delta \tag{5.8}$$

by mapping $\infty \mapsto [\Delta]$. This map is also $(2N-3)$-connected.

As K, we can take the space $G \cap B(0, r_k) - \mathrm{int}B(0, r_0)$ in the definition of $E_\bullet$. This leads to the following definition.

Definition 5.17. Let $\tilde{E}_\bullet$ be the simplicial space where an element of $\tilde{E}_k$ is a pair (χ, f), where $\chi = (G, r_0 < r_1 < \cdots < r_k, \{\varphi_i\}) \in N_k\mathscr{C}$ and f is an element

$$f \in \mathrm{Map}(K, S^{N-1})/\Delta,$$

where $K = G \cap B(0, r_k) - \mathrm{int}B(0, r_0)$ and Δ denotes the subset of constant maps.

The subset K in the above definition will be a *forest*, i.e. a disjoint union of (at least two) contractible graphs. We should explain the topology on the space $\tilde{E}_k$. The main observation is that if $\chi, \chi' \in N_k\mathscr{C}$ and K, K' are the corresponding forests, then there will be a canonical map $\varphi : K' \to K$ whenever χ' is sufficiently close to χ. (By the definition of the topology on $\Phi(\mathbb{R}^N)$, any G' near G will admit a map $\tilde{\varphi} : G' \dashrightarrow G$ whose domain contains K' and whose image contains K. After reparametrizing edges it will restrict to a map from K' onto K.) We topologize $\tilde{E}_k$ by declaring (χ', f') close to (χ, f) if χ' is close to χ and f' is close to $f \circ \varphi$.

For the following lemma, recall the notion of *fiber homotopy* from [Dol63], and some related notions. If $f : E \to B$ and $f' : E' \to B$ are two maps, then a *fiber homotopy* is a homotopy $F : E \times [0,1] \to E'$ over B. A map $g : E \to E'$ over B is a *fiber homotopy equivalence* if it admits a map $h : E' \to E$ which is left and right inverse to g up to fiber homotopy. A map $E \to B$ is *fiber homotopy trivial* if it is fiber homotopy equivalent to a projection $B \times F \to B$. A map $f : E \to B$ is *locally fiber homotopy trivial* if B admits a covering by open sets U such that the restriction $f^{-1}(U) \to U$ is fiber homotopy trivial. It is shown in [Dol63, theorem 6.4] that local fiber homotopy triviality is sufficient for the "long exact sequence for a fibration": if $f : E \to B$ is locally fiber homotopy trivial, then the homotopy groups of a fibers $F_b = f^{-1}(b)$ fit into a long exact sequence with $\pi_*(E)$ and $\pi_*(B)$.

It follows from the definition that the projection $\tilde{E}_k \to N_k\mathscr{C}$ is locally fiber homotopy trivial. Indeed, let $U \subseteq N_k\mathscr{C}$ be a neighborhood of χ small enough that any $\chi' \in U$ admits a canonical map $\varphi : K' \to K$ (cf. the discussion following definition 5.17). We get a map

$$U \times \mathrm{Map}(K, S^{N-1}) \to \tilde{E}_k,$$

given by $(\chi', f) \mapsto (\chi', f \circ \varphi)$, which restricts to a fiber homotopy equivalence over U.

Lemma 5.18. *The map A above induces N+highly connected maps $|E_\bullet| \to |\tilde{E}_\bullet|$ and $|E_\bullet|/B\mathscr{C} \to |\tilde{E}_\bullet|/B\mathscr{C}$.*

Proof. Both maps $E_k \to N_k\mathscr{C}$ and $\tilde{E}_k \to N_k\mathscr{C}$ induce long exact sequences in homotopy groups. For $\tilde{E}_k$, this was explained above, and for E_k it can be proved in the following way. We shall prove later (lemma 5.21) that $N_k\mathscr{C}$ has trivial π_1 and π_2. A similar argument shows that E_k is simply connected. Hence the homotopy fiber of the projection $E_k \to N_k\mathscr{C}$ is simply connected. From [MS76, proposition 5] it follows that the inclusion of the fiber of the projection $E_k \to N_k\mathscr{C}$ into the homotopy fiber is a *homology equivalence*. Since both the homotopy fiber and the fiber are simply connected, it is actually a homotopy equivalence, so $E_k \to N_k\mathscr{C}$ is a quasifibration.

The induced map on fibers is the map (5.8), so the first part of the lemma follows from the 5-lemma. The second map uses that the inclusions of $B\mathscr{C}$ into $|E_\bullet|$ and $|\tilde{E}_\bullet|$ are cofibrations. □

An element of $N_l(\mathscr{C} \wr (H \circ T))$, where T is the functor from definition 5.14, is given by an element $(G, r_0 < r_1 < \cdots < r_l, \{\varphi_i\}) \in N_l(\mathscr{C})$ together with an element $g \in \mathrm{Map}(\underline{k_0}, S^{N-1})/\Delta$. Here,

$$\varphi_i : \underline{k_i} \to G \cap \partial B(0, r_i)$$

are the labellings. Again, let $K = G \cap B(0, r_l) - \mathrm{int}B(0, r_0)$. The labelling φ_0 in the first vertex induces an injective map

$$\varphi_0 : \underline{k_0} \to K$$

which is a homotopy equivalence. It has a unique left inverse which we denote φ_0^{-1}. Up to homotopy φ_0^{-1} is also right inverse to φ_0.

Composition with φ_0^{-1} induces a homotopy equivalence

$$\mathrm{Map}(k_0, S^{N-1})/\Delta \xrightarrow{\circ\varphi_0^{-1}} \mathrm{Map}(K, S^{N-1})/\Delta$$

and in turn a simplicial map

$$N_\bullet(\mathscr{C} \wr (H \circ T)) \to \tilde{E}_\bullet \tag{5.9}$$

which is a degreewise homotopy equivalence. Similarly to lemma 5.18, this proves the following lemma.

Lemma 5.19. *The maps*

$$B(\mathscr{C} \wr (H \circ T)) \to |\tilde{E}_\bullet|,$$
$$B(\mathscr{C} \wr (H \circ T))/B\mathscr{C} \to |\tilde{E}_\bullet|/B\mathscr{C}$$

induced by (5.9) *are weak homotopy equivalences.* □

Combining proposition 5.4 and lemmas 5.18 and 5.19, we get the following.

Corollary 5.20. *There is an N+highly connected map $\Phi(\mathbb{R}^N) \to B(\mathscr{C} \wr (H \circ T))/B\mathscr{C}$.* □

Corollary 5.20 states that stably (i.e. for $N \to \infty$), we can regard $\Phi(\mathbb{R}^N)$ as the pointed homotopy colimit of the functor $(H \circ T)$ over the topological category $\mathscr{C}$. We would like to replace that with the pointed homotopy colimit of the functor H over the category $\mathscr{D}_{\geq 2}$, whose objects are finite sets $\underline{k}$ of cardinality at least 2 and whose morphisms are surjections.

Lemma 5.21. *The functor $T : \mathscr{C} \to \mathscr{D}_{\geq 2}^{\mathrm{op}}$ induces a highly connected map $N_l T : N_l\mathscr{C} \to N_l\mathscr{D}_{\geq 2}^{\mathrm{op}}$ for all l.*

Proof. The codomain $N_l\mathscr{D}_{\geq 2}$ is a discrete set. Let $(\underline{k_0} \to \underline{k_1} \to \cdots \to \underline{k_l}) \in N_l\mathscr{D}_{\geq 2}^{\mathrm{op}}$. A point in the inverse image is given by embeddings of the finite sets $\underline{k_i}$ into $(N-1)$-spheres, and trees with these sets as the set of leaves. Embeddings of finite sets into an $(N-1)$-sphere form an $(N-3)$-connected space. Trees with a fixed set of leaves form an $(N-4)$-connected space by theorem 3.19 (applied with $M = B(0, a_j) - \mathrm{int}B(0, a_{j-1})$, and using that A_0^s is the trivial group). □

The approximation in lemma 5.21 may seem to be not good enough. $\Omega^\infty\mathbf{\Phi}$ is the direct limit of the spaces $\Omega^N\Phi(\mathbb{R}^N)$, so we should deal with spaces up to N+highly connected maps instead of just up to highly connected maps. Surprisingly, the extra N comes for free. (Analogously, if $f : X \to Y$ is c-connected and ξ is an N-dimensional vector bundle over Y, then the map of Thom spaces $X^{f^*\xi} \to Y^\xi$ is $(c+N)$-connected.) Proposition 5.22 finishes the proof of proposition 5.16.

Proposition 5.22. *The map*

$$B(\mathscr{C} \wr (H \circ T))/B\mathscr{C} \to B(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr H)/B\mathscr{D}^{\mathrm{op}}_{\geq 2}$$

is N+highly connected.

Proof. For all k we have the following pullback diagram.

$$\begin{tikzcd}
N_k(\mathscr{C} \wr (H \circ T)) \arrow[r] \arrow[d] & N_k(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr H) \arrow[d] \\
N_k\mathscr{C} \arrow[r] & N_k\mathscr{D}^{\mathrm{op}}_{\geq 2}.
\end{tikzcd}$$

The right hand vertical map is a fibration, so the diagram is also homotopy cartesian. Both vertical maps are split, using the canonical basepoint $\infty \in H$. It follows that the diagram

$$\begin{tikzcd}
N_k\mathscr{C} \arrow[r] \arrow[d] & N_k\mathscr{D}^{\mathrm{op}}_{\geq 2} \arrow[d] \\
N_k(\mathscr{C} \wr (H \circ T)) \arrow[r] & N_k(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr H)
\end{tikzcd}$$

is also homotopy cartesian (horizontal homotopy fibers are homotopy equivalent).

The vertical and horizontal maps are all $(N-3)$-connected. It follows by the Blakers-Massey theorem that the diagram is $(N-3)+(N-3)-1 = (2N-7)$-cocartesian. This means precisely that the induced map of vertical cofibers is $(2N-7)$-connected and the claim follows. □

Thus, we have an N+highly connected map from $\Phi(\mathbb{R}^N)$ to the pointed homotopy colimit of the functor $H : \mathscr{D}^{\mathrm{op}}_{\geq 2} \to \mathrm{Spaces}$. We proceed to determine the homotopy type of this pointed homotopy colimit. Recall that $H(\underline{k}) = \mathrm{Map}(\underline{k}, S^{N-1})/\Delta$. The pointed homotopy colimit is homeomorphic to the quotient

$$B(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr \mathrm{Map}(-, S^{N-1}))/B(\mathscr{D}^{\mathrm{op}}_{\geq 2} \wr \Delta) \tag{5.10}$$

where Δ denotes the constant functor S^{N-1}.

Proposition 5.23. *The spaces $B(\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr \mathrm{Map}(-, S^{N-1}))$ and $B\mathscr{D}_{\geq 2}$ are both contractible.*

Proof of theorem 5.1. Proposition 5.23 implies that $B(\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr \Delta) \cong B\mathscr{D}_{\geq 2}^{\mathrm{op}} \times S^{N-1} \simeq S^{N-1}$, so the quotient in (5.10) becomes S^N and we get an $N+$highly connected map

$$\Phi(\mathbb{R}^N) \to S^N \tag{5.11}$$

The map (5.11) is a zig-zag of $N+$highly connected maps, all of which induce spectrum maps as N varies. It follows that there is a weak equivalence of spectra $\mathbf{\Phi} \simeq S^0$ as claimed. □

Remark 5.24. For an object $\underline{k} \in \mathscr{D}_{\geq 2}$, let $\Delta \to (S^{-1})^{\underline{k}}$ be the inclusion of the diagonal into the k-fold power of the spectrum S^{-1}. Let $(S^{-1})^{\underline{k}}/\Delta$ be the cofiber. Then we have proved two homotopy equivalences

$$\mathbf{\Phi} \simeq \operatorname*{hocolim}_{\underline{k} \in \mathscr{D}_{\geq 2}} \left((S^{-1})^{\underline{k}}/\Delta \right) \simeq S^0.$$

Proof of proposition 5.23. We have a functor $\mathscr{D}_{\geq 2} \to \mathscr{D}_{\geq 2}$ given by $T \mapsto \underline{2} \times T$, and the projections define natural transformations

$$T \longleftarrow \underline{2} \times T \longrightarrow \underline{2}.$$

This contracts $B\mathscr{D}_{\geq 2}$ to the point $\underline{2} \in B\mathscr{D}_{\geq 2}$.

For the space

$$B(\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr \mathrm{Map}(-, S^{N-1})) = \operatorname*{hocolim}_{T \in \mathscr{D}_{\geq 2}^{\mathrm{op}}} \mathrm{Map}(T, S^{N-1})$$

we use a trick strongly inspired by the works of [Han00] and [BD04, §3.4.1], which prove that the colimit (not homotopy colimit) is contractible.

Choose a (symmetric monoidal) disjoint union functor $\amalg : \mathscr{D}_{\geq 2} \times \mathscr{D}_{\geq 2} \to \mathscr{D}_{\geq 2}$. For brevity, denote the functor $\mathrm{Map}(-, S^{N-1})$ by J. The disjoint union functor induces a functor

$$(\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr J) \times (\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr J) \to (\mathscr{D}_{\geq 2}^{\mathrm{op}} \wr J)$$

which is associative and commutative up to natural transformation. It follows that the classifying space is a homotopy associative and homotopy commutative H-space.

In this H-space structure, multiplication by 2 is homotopic to the identity. This follows from the natural transformation $T \amalg T \to T$. The claim then follows from lemma 5.25 below. □

Lemma 5.25. *A connected, homotopy associative, homotopy commutative H-space X is weakly contractible if multiplication by 2 (i.e. the map $x \mapsto x \cdot x$) is homotopic to the identity.*

This lemma is completely trivial when X has a homotopy unit. In that case, it is well know that the map induced by the H-space structure

$$\pi_* X \times \pi_* X \to \pi_* X$$

agrees with the usual group multiplication on homotopy groups. Hence all $x \in \pi_* X$ satisfies $x + x = x$. The proof in the general case is a variation of this argument.

Proof. Let $\mu : X \times X \to X$ be the H-space structure. Choose a basepoint $x_0 \in X$ and write $\pi_n(X) = \pi_n(X, x_0)$. We can assume that μ is a pointed map. The two projections $p, q : X \times X \to X$ induce an isomorphism

$$(p_*, q_*) : \pi_n(X \times X) \to \pi_n X \times \pi_n X$$

and we let

$$\bullet = \mu_* \circ (p_*, q_*)^{-1} : \pi_n X \times \pi_n X \to \pi_n X.$$

This is now an associative, commutative product on $\pi_n X$ satisfying $x \bullet x = x$ for all $x \in \pi_n(X)$. Let $+$ denote the usual group structure on $\pi_n X$ and write 0 for the identity element with respect to $+$ (we will write it additively although we don't yet know that it is commutative for $n = 1$).

Let $\Delta : X \to X \times X$ be the diagonal and $i, j : X \to X \times X$ the inclusions $i(x) = (x, x_0)$, $j(x) = (x_0, x)$. Then we have

$$\begin{aligned}(p_*, q_*) \circ (i_* + j_*)(x) &= ((p \circ i)_*, (q \circ i)_*)(x) + ((p \circ j)_*, (q \circ j)_*)(x) \\ &= (x, 0) + (0, x) = (x, x) = (p_*, q_*) \circ \Delta_*(x).\end{aligned}$$

It follows that $\Delta_* = i_* + j_*$ because (p_*, q_*) is an isomorphism. Now $\mu \circ \Delta \simeq \mathrm{id}$ and $\mu \circ i \simeq \mu \circ j$, so

$$x = \mu_* \Delta_*(x) = \mu_* i_* x + \mu_* j_* x = 2x \bullet x_0.$$

Substituting $x \bullet x_0$ for x then gives

$$x \bullet x_0 = 2x \bullet x_0 \bullet x_0 = 2x \bullet x_0 = x$$

which in turn gives that $x = 2x$ for any $x \in \pi_n X$. It follows that X is weakly contractible. $\square$

6. Remarks on manifolds

Most of the results of this paper works equally well for the sheaf Ψ_d, where $\Psi_d(U)$ is the space of all closed sets $M \subseteq U$ which are smooth d-dimensional submanifolds without boundary. A neighborhood basis at M is formed by the sets

$$\mathcal{V}_{K,W} = \{N \in \Psi_d(U) | N \cap K = j(M) \cap K \text{ for some } j \in W\},$$

where $K \subseteq U$ is a compact set and $W \subseteq \mathrm{Emb}(M, U)$ is a neighborhood of the inclusion in the Whitney C^∞ topology.

The analogues of propositions 4.8 and 4.9 hold with almost identical proofs and give the following weak equivalence.

$$B\mathcal{C}_d^N \simeq \Omega^{N-1}\Psi_d(\mathbb{R}^N) \tag{6.1}$$

Here $\mathcal{C}_d^N$ is the cobordism category whose objects are closed $(d-1)$-manifolds $M \subseteq \{a\} \times \mathbb{R}^{N-1}$ and whose morphisms are compact d-manifolds $W \subseteq [a_0, a_1] \times \mathbb{R}^{N-1}$, cf. [GMTW06, section 2]. We note that for the proof of lemma 4.13ii we will no longer necessarily have $D_{N,k}$ connected; however it will be a grouplike topological monoid, which suffices for the proof.

Let $\mathrm{Gr}_d(\mathbb{R}^N)$ be the Grassmannian of d-planes in $\mathbb{R}^N$, and $U_{d,N}^\perp$ the canonical $(N-d)$-dimensional vector bundle over it. A point in $U_{d,N}^\perp$ is given by a pair $(V, v) \in \mathrm{Gr}_d(\mathbb{R}^N) \times \mathbb{R}^N$ with $v \perp V$. Let

$$q : U_{d,N}^\perp \to \Psi_d(\mathbb{R}^N)$$

be the map given by $q(V, v) = V - v \in \Psi_d(\mathbb{R}^N)$. This gives a homeomorphism onto the subspace of manifolds $M^d \subseteq \mathbb{R}^N$ which are affine subspaces. q extends continuously to the one-point compactification of $U_{d,N}^\perp$ by letting $q(\infty) = \emptyset$. This one-point compactification is the *Thom space* $\mathrm{Th}(U_{d,N}^\perp)$, and we get a map

$$q : \mathrm{Th}(U_{d,N}^\perp) \to \Psi_d(\mathbb{R}^N). \tag{6.2}$$

We will show that (6.2) is a weak equivalence. Define two open subsets $U_0 \subseteq \Psi_d(\mathbb{R}^N)$ and $U_1 \subseteq \Psi_d(\mathbb{R}^N)$ in the following way. U_0 is the space of d-manifolds M such that $0 \notin M$, and U_1 is the space of manifolds such that the function $p \mapsto |p|^2$ has a unique, non-degenerate minimum on M. Let $U_{01} = U_0 \cap U_1$. These are open subsets, and $\Psi_d(\mathbb{R}^N)$ is the pushout of $(U_0 \leftarrow U_{01} \to U_1)$.

Lemma 6.1. *Each restriction of q*

$$q^{-1}(U_0) \to U_0$$
$$q^{-1}(U_{01}) \to U_{01}$$
$$q^{-1}(U_1) \to U_1$$

is a weak homotopy equivalence. Consequently (6.2) is a weak equivalence.

Proof. U_0 and $q^{-1}(U_0)$ are both contractible: $q^{-1}(U_0)$ contracts to the point ∞, and the path constructed in the proof of lemma 2.6 gives a contraction of U_0, pushing everything to infinity, radially away from 0.

For U_1 a deformation retraction is defined as follows. Let $M \in U_1$ have p as unique minimum of $p \mapsto |p|^2$. Let $\varphi_t(x) = p + (1-t)(x-p)$. This defines a diffeomorphism $\mathbb{R}^N \to \mathbb{R}^N$ for $t < 1$. A path γ in U_1 from M to a point in the image of q is defined by $\gamma(t) = \varphi_t^{-1}(M)$ for $t < 1$ and $\gamma(1) = p + T_pM$. This proves that $q^{-1}(U_1) \to U_1$ is a deformation retraction. This deformation restricts to a deformation retraction of $q^{-1}(U_{01}) \to U_{01}$. □

We have proved the following result.

Proposition 6.2. $q : \mathrm{Th}(U_{d,N}^\perp) \to \Psi_d(\mathbb{R}^N)$ *is a weak equivalence.* □

Thus we have proved the following theorem. In the limit $N \to \infty$ we recover the main theorem of [GMTW06], but theorem 6.3 holds also for finite N.

Theorem 6.3. *There is a weak homotopy equivalence*

$$B\mathcal{C}_d^N \simeq \Omega^{N-1}\mathrm{Th}(U_{d,N}^\perp).$$ □

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